Wedge-local observables in integrable quantum models with bound states

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PhD Thesis

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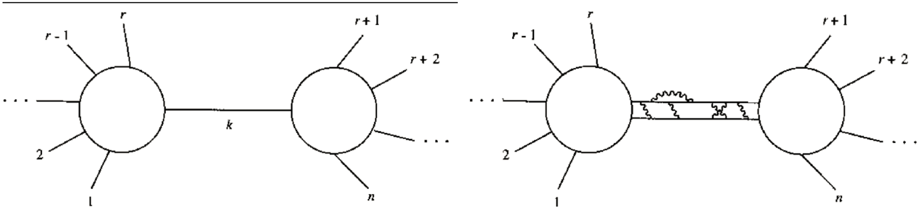
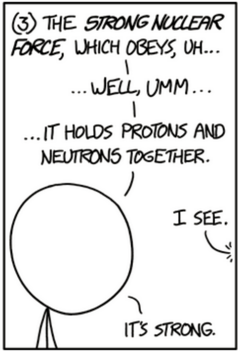
Motivation: Bound states in QFT

- Make up the world (proton/neutron)
- Non-perturbative phenomenon / Described by effective theories
- Energy of the bound state below the unbound states
- Correspond to poles in the S-matrix



[Weinberg, *Quantum Theory of Fields 1*]



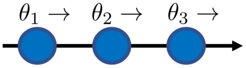
Integrable quantum models (IQMs)

Lets look at a simple class of interacting QFTs

Integrable Quantum models:

- 1+1 dimensional QFTs
- Interacting
- Fock-space structure $z, z^\dagger$
- Non-perturbatively known S-matrix

Rapidity states: $|\theta_1, \theta_2, \theta_3\rangle =$



Interesting from the perspective of AQFT:

- Operator algebras $\mathcal{O} \mapsto \mathcal{A}(\mathcal{O})$ constructed [Lechner '08], [Lechner/Alazzawi '17]
- Starting point: Wedge-algebras $\mathcal{A}(W_L + x)$, $\mathcal{A}(W_R + y)$

Models (selection)	one particle species	several particles species
No bound states	Massive Ising, Sinh-Gordon	$O(N)$-invariant sigma
With bound states	Bullough-Dodd	$Z(N)$-Ising

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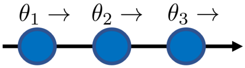

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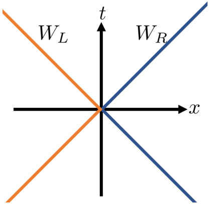

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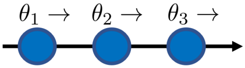

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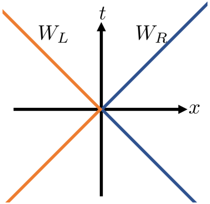

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The inverse scattering approach

Integrable quantum models are defined by the S-matrix (no Lagrangian needed)

$S : \mathbb{R} + i[0,\pi] \to \mathbb{C}$ analytic (up to isolated poles)

Hilbert space: S-symmetrized Fock space

$$S(\theta_1 - \theta_2)\Psi_n(\theta_1, \theta_2, ..) = \Psi_n(\theta_2, \theta_1, ..),$$

$$z^\dagger(\theta_1) z^\dagger(\theta_2) = S(\theta_1 - \theta_2) z^\dagger(\theta_2) z^\dagger(\theta_1)$$

Bound state structure: poles in the S-matrix

$$p_\alpha(\theta + i\theta_{(\alpha\beta)}) + p_\beta(\theta - i\theta_{(\beta\alpha)}) = p_\gamma(\theta),$$

$$\underset{\theta=0}{\mathrm{Res}}\ S_{\alpha\beta}\Big(\theta + i[\theta_{(\alpha\beta)} + \theta_{(\beta\alpha)}]\Big) = R^\gamma_{\alpha\beta}$$

$$\left[\text{Note: } p_\alpha(\theta) = \begin{pmatrix} m_\alpha \cosh\theta \\ m_\alpha \sinh\theta \end{pmatrix}\right]$$

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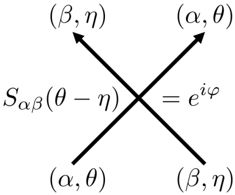

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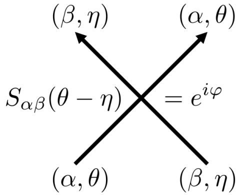

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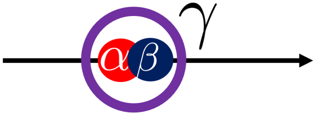

Characterization of wedge-local observables

The wedge-local fields

In IQMs without bound states, the "free field" equivalent is wedge-local: [Lechner '08]

$$\text{Left-wedge field: } \phi(f) = \int d\theta \Big(f^+(\theta) z^\dagger(\theta) + f^-(\theta) z(\theta) \Big),$$

$$f^\pm(\theta) = \int d^2x\, f(x) e^{\pm i p(\theta)\cdot x}$$

$$\text{Right-wedge field: } \phi'(g) = J\phi(Jg)J \leftarrow \text{Modular conjugation},$$

Wedge-locality condition:

$$[\phi(f), \phi'(g)] = 0 \text{ for } f \in \mathcal{D}(W_L),\ g \in \mathcal{D}(W_R)$$

Generators of wedge-algebras

$$\mathcal{A}(W_L) := \{e^{i\phi(f)} : f \in \mathcal{D}(W_L)\}'' \qquad \mathcal{A}(W_R) := \{e^{i\phi'(g)} : g \in \mathcal{D}(W_R)\}''$$

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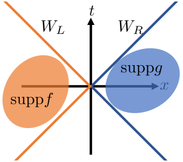

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The Araki Expansion

Fock Space structure $\rightarrow A = \sum_{m,n\in\mathbb{N}_0} \int \frac{d^m\boldsymbol{\theta} d^n\boldsymbol{\eta}}{m!n!} f_{m,n}(\boldsymbol{\theta}|\boldsymbol{\eta})\, z^{\dagger m}(\boldsymbol{\theta}) z^n(\boldsymbol{\eta})$

↑ connected matrix elements

Matrix elements without Dirac delta terms:

$$\langle z^\dagger(\theta_1) z^\dagger(\theta_2)\Omega | A z^\dagger(\eta)\Omega\rangle =$$

$$f_{2,1}(\theta_1,\theta_2|\eta) \quad + \quad f_{1,0}(\theta_1)\delta(\theta_2-\eta) \quad + \quad f_{1,0}(\theta_2)S(\theta_1-\theta_2)\delta(\theta_1-\eta)$$

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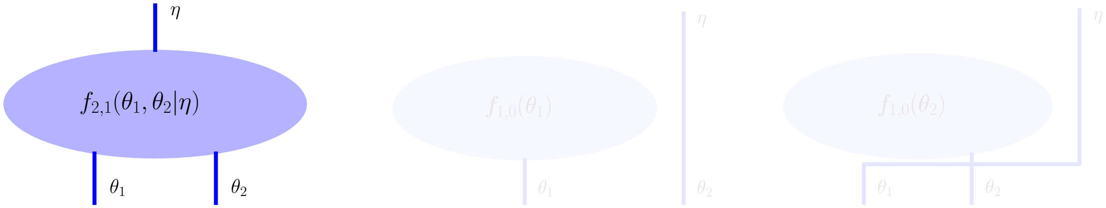

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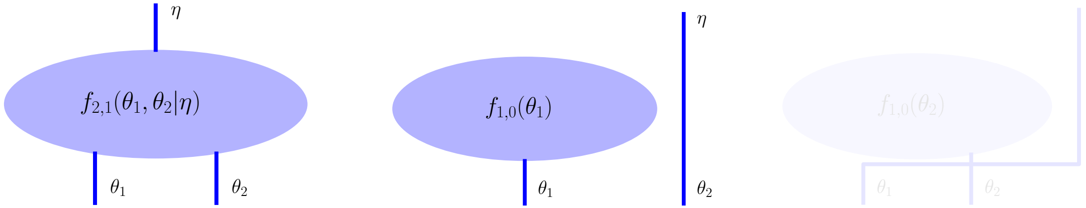

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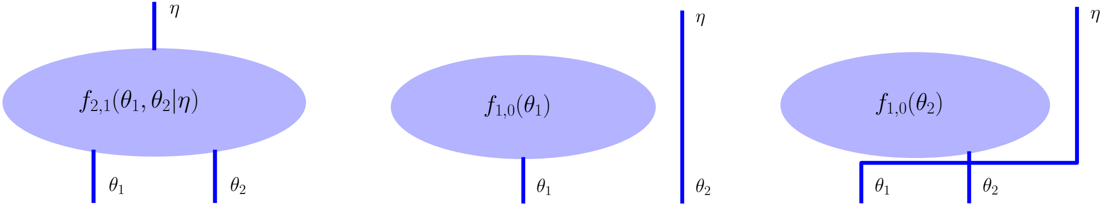

The characterization theorem

Theorem (Characterization theorem: [Bostelmann/Cadamuro '15])

A wedge-local observable A (in the sense that $[A, \phi'(g)] = 0$) satisfies

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where $F_k : \mathcal{I}_+^k \to \mathbb{C}$ is analytic and satisfies a list of properties.

- $f_{m,n} \to F_{m+n}$ (many matrix elements are boundary values of the same function).
- F_k highly restricted (analyticity, additional properties)

Examples: (Powers of) the wedge-field.

$$\boxed{A = \phi(f)} \quad F_1(\theta) = f^+(\theta), \; F_1(\eta + i\pi) = f^-(\eta) \; , \quad F_k = 0 \text{ for } k \neq 1,$$

Properties of F_k for local observables are well established (Bootstrap program) [Babujian/Karowski '04].

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IQMs with elementary bound states

Wedge-local quantities in IQMs with bound states

$\phi(f)$ not wedge-local anymore! Has to be modified by a bound state operator: [Cadamuro/Tanimoto '15, '17]

$$\widetilde{\phi}(f) = \phi(f) + \chi(f),$$

$$[\chi(f)\Psi]_1^\gamma(\theta) = \sum_{\alpha,\beta} \sqrt{2\pi|R^\gamma_{\alpha\beta}|} f^+_\alpha(\theta + i\theta_{(\alpha\beta)})\Psi_1^\beta(\theta - i\theta_{(\beta\alpha)}).$$

Formal expression $\chi(f) = \sum_{\alpha,\beta,\gamma} \int d\theta \; f^+_\alpha(\theta + i\theta_{(\alpha\beta)}) z^\dagger_\gamma(\theta) z_\beta(\theta - i\theta_{(\beta\alpha)})$

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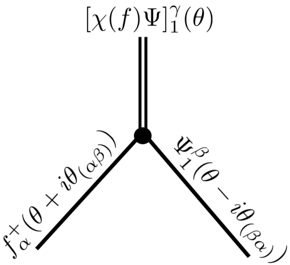

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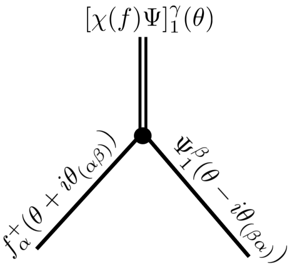

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with F_k as before, except for additional bound state poles

$$2\pi i \underset{\theta_2 = \theta_1 + i\frac{2\pi}{3}}{Res} F_k(\theta_1, \theta_2, \theta_3, \ldots) = -\sqrt{2\pi|R|} F_{k-1}\left(\theta_1 + i\frac{\pi}{3}, \theta_3, \ldots\right)$$

Implications:

- Recursion relation: $F_k \xrightarrow{\text{Res}} F_{k-1} \xrightarrow{\text{Res}} F_{k-2} \cdots$
  $\Rightarrow$ Araki expansion of wedge-local observables never terminates (infinite orders)!
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Sketch of the proof

$$\langle \Phi_m, [A, \phi'(g)]\Psi_n\rangle \sim \int d\xi \, \Big(g^-(\xi) F_{m+n+1}(\boldsymbol{\theta}, \xi, \boldsymbol{\eta} + i\boldsymbol{\pi}) - g^-(\xi + i\pi) F_{m+n+1}(\boldsymbol{\theta}, \xi + i\pi, \boldsymbol{\eta} + i\boldsymbol{\pi}) \Big)$$

Rewrite this as a contour integral

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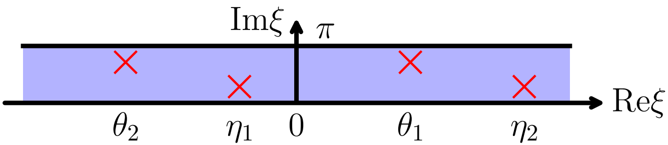

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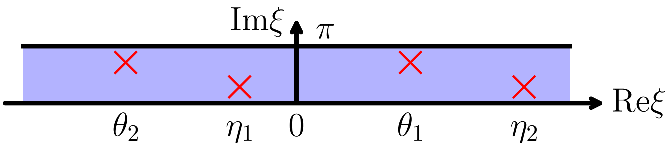

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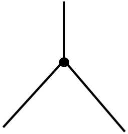
IQMs with composite bound states

Models with “higher-order” bound state structure



In scalar IQMs with bound states, the structure is simple:
No composite particles!

The scaling Z(4)-Ising model:
Two elementary particles $\{1, \bar{1}\}$ and one composite particle $\{2 = \bar{2}\}$.

Higher-order bound state structure
⇒ **Double poles** in the S-matrix [Coleman/Thun '74]

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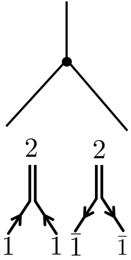
$$\operatorname*{Res}_{\theta_2=\theta_1+i\frac{\pi}{2}} F_k^{22\cdots}(\theta_1,\theta_2,\ldots) = 2\pi|R|F_k^{1\bar{1}\cdots}\left(\theta_1+i\frac{\pi}{4},\theta_1+i\frac{\pi}{4},\ldots\right)$$

Models with "higher-order" bound state structure



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The scaling Z(4)-Ising model:
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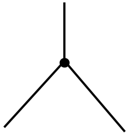

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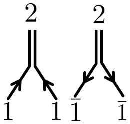

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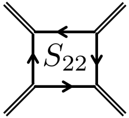

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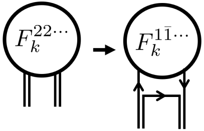

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A reversal of strategies

In models with elementary bound state structure:

Have: Wedge-local field $\phi(f) + \chi(f)$.
Want: Proof of form factor properties F_k.

$$\boxed{[A, \phi'(g) + \chi'(g)] = 0} \Rightarrow \boxed{F_k}$$

In models with composite bound state structure:

Wedge-local field generally unknown
$\phi(f) + \chi(f) + X(f)$?. [Cadamuro/Tanimoto '17]

Idea: Use expected form factor properties as axioms → obtain a candidate for the wedge-local field.

Have: Form Factor Axioms (Bootstrap program [Babujian/Foerster/Karowski '06])
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A second-order bound state operator in Z(4)-Ising [PhD Results]

Conjecture (Wedge-local field in the Z(4)-Ising model)

Given the form factor axioms in the Z(4)-Ising model, the wedge-local field (satisfying $[A, \widetilde{\phi}'(g)]$) takes the form

$$\widetilde{\phi}(f) = \phi(f) + \chi(f) + X(f),$$

where $X(f)$ is the second-order bound state operator, given by

$$X(f) = 2\pi |R| \int d\theta \left[f_2^+ \left(\theta + i\frac{\pi}{2}\right) z_2^\dagger(\theta) z_1 \left(\theta - i\frac{\pi}{4}\right) z_{\bar{1}} \left(\theta - i\frac{\pi}{4}\right) + h.c. \right]$$

Problems:

- $X(f)$ is not an operator on the Hilbert space (bounded or unbounded), since $[X(f)\Psi](\theta_1, \theta_2) \propto \delta(\theta_1 - \theta_2)$.
- $[X(f), X'(g)]$ cannot be computed, because product $X(f)X'(g)$ is ill-defined.
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Conclusion

What have we learned?

Simple bound state structure

Theorem: Characterization of wedge-local observables via form factors

$$A = \sum_{m,n\in\mathbb{N}_0} F_{m+n}(z^{\dagger m} z^n)$$

Insights:

Expansion is always infinite: $F_k \neq 0\ \forall k \in \mathbb{N}_0$.

Class of expansion distinct from wedge-local field,

$$\widetilde{\phi}(f) = \phi(f) + \chi(f),$$
$$\chi(f) \propto z^\dagger_\gamma(\theta) z_\beta(\theta - i\theta_{(\beta\alpha)})$$

(Quadratic in $z^\dagger z$, contains complex shift)

Models with composite particles

Conjecture: The wedge-local field has to be supplemented by a "second-order bound state operator",

$$\widetilde{\phi}(f) = \phi(f) + \chi(f) + X(f).$$

Insights:

More intricate bound state structure
$\Rightarrow$ more intricate structure of wedge-local observables.

$X(f)$ ill-defined as an operator
$\Rightarrow$ not the right tool for construction of wedge-algebras

$$\mathcal{A}(W_L) := \{e^{i\widetilde{\phi}(f)} : f \in \mathcal{D}(W_L)\}''$$

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