GREEN FUNCTIONS OF KLEIN GORDON EQUATION ON CURVED SPACETIMES

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On many spacetimes one can define four natural Green functions of the Klein-Gordon equation:

- forward propagator $G^{\vee}$,
- backward propagator $G^{\wedge}$,
- Feynman propagator G^{F},
- antiFeynman propagator $G^{\overline{\mathrm{F}}}$.

In some rare but important cases they satisfy the identity

$$G^{\mathrm{F}} + G^{\overline{\mathrm{F}}} = G^{\vee} + G^{\wedge}.$$

We will then say that the Klein-Gordon equation is *special*. We will discuss consequences of this property and describe examples of special spacetimes.

PART I. FLAT SPACETIME.

Consider first the Klein-Gordon equation on the flat *Minkowski space* $\mathbb{R}^{1,d-1}$ wth $m^2 \geq 0$:

$$(-\Box + m^2)\psi = 0.$$

$G(x, y)$ is a *Green function of the Klein Gordon equation* if

$$(-\Box_x + m^2)G(x, y) = \delta(x - y).$$

There are four Green functions invariant wrt the restricted Poincaré group:

- the *forward/backward propagator*

$$G^{\vee/\wedge}(x,y) := \frac{1}{(2\pi)^4}\int \frac{\mathrm{e}^{-\mathrm{i}(x-y)\cdot p}}{p^2+m^2 \pm \mathrm{i}0\,\mathrm{sgn}\,p_0}\,\mathrm{d}p,$$

- the *Feynman/anti-Feynman propagator*

$$G^{\mathrm{F}/\overline{\mathrm{F}}}(x,y) := \frac{1}{(2\pi)^4}\int \frac{\mathrm{e}^{-\mathrm{i}(x-y)\cdot p}}{p^2+m^2 \mp \mathrm{i}0}\,\mathrm{d}p.$$

Green functions $G^{\vee}$ and $G^{\wedge}$ are related to the classical Cauchy problem, because their support is in the forward, resp. backward cone. Green functions G^{F} and $G^{\overline{\mathrm{F}}}$ are used in QFT.

They satisfy the identity $G^{\mathrm{F}} + G^{\overline{\mathrm{F}}} = G^{\vee} + G^{\wedge}$.

Using the above Green functions we can define the following useful *bisolutions of the Klein-Gordon operator*:

- the *Pauli–Jordan propagator* or *commutator function*

$$G^{\mathrm{PJ}}(x,y) \coloneqq G^{\vee} - G^{\wedge},$$

- the *positive frequency* or *Wightman* 2-point function

$$G^{(+)}(x,y) \coloneqq \frac{1}{\mathrm{i}}(G^{\mathrm{F}} - G^{\wedge}) = \frac{1}{\mathrm{i}}(-G^{\overline{\mathrm{F}}} + G^{\vee}),$$

- the *negative frequency* or *anti-Wightman* 2-point function

$$G^{(-)}(x,y) \coloneqq \frac{1}{\mathrm{i}}(-G^{\overline{\mathrm{F}}} + G^{\wedge}) = \frac{1}{\mathrm{i}}(G^{\mathrm{F}} - G^{\vee}).$$

The following facts are easy to see:

(1) The Klein-Gordon operator $K = -\Box + m^2$ is *essentially self-adjoint* on $C_c^\infty(\mathbb{R}^{1,3})$ in the sense of $L^2(\mathbb{R}^{1,3})$.
(2) For $s > \frac{1}{2}$, as an operator $\langle t\rangle^{-s} L^2(\mathbb{R}^{1,3}) \to \langle t\rangle^{s} L^2(\mathbb{R}^{1,3})$, the Feynman propagator is the *boundary value of the resolvent of the Klein-Gordon operator*:

$$\operatorname*{s-lim}_{\epsilon \searrow 0} (K \mp \mathrm{i}\epsilon)^{-1} = G^{\mathrm{F}/\overline{\mathrm{F}}}.$$

Here $\langle t\rangle$ denotes the so-called "Japanese bracket"

$$\langle t\rangle := \sqrt{1+t^2}.$$

After *quantization*, we obtain an operator–valued distribution $\mathbb{R}^{1,d-1} \ni x \mapsto \psi^*(x) = \psi(x)^*$ satisfying the Klein-Gordon equation and commutation relations

$$(-\Box + m^2)\psi^*(x) = 0,$$
$$[\hat{\psi}(x), \hat{\psi}^*(y)] = -\mathrm{i}G^{\mathrm{PJ}}(x, y).$$

We also have a representation with the state given $(\Omega| \cdot \Omega)$ such that

$$(\Omega \,|\, \hat{\psi}(x)\hat{\psi}^*(y)\Omega) = G^{(+)}(x, y),$$
$$(\Omega \,|\, \hat{\psi}^*(x)\hat{\psi}(y)\Omega) = G^{(-)}(x, y).$$

$$\big(\Omega \,\big|\, \mathrm{T}\big(\hat{\psi}(x)\hat{\psi}^*(y)\big)\Omega\big) = -\mathrm{i}G^{\mathrm{F}}(x, y),$$
$$\big(\Omega \,\big|\, \overline{\mathrm{T}}\big(\hat{\psi}(x)\hat{\psi}^*(y)\big)\Omega\big) = \mathrm{i}G^{\overline{\mathrm{F}}}(x, y).$$

PART II. CURVED SPACETIMES.

Consider a curved spacetime M with the *metric tensor* $g_{\mu\nu}$. Define the *d'Alembertian* and the *Klein-Gordon operator*

$$-\Box := -|g|^{-\frac{1}{2}}\partial_\mu |g|^{\frac{1}{2}} g^{\mu\nu}\partial_\nu, \qquad K := -\Box + m^2.$$

(One could also replace the term m^2 with a scalar potential). How to generalize the well-known propagators from $\mathbb{R}^{1,d-1}$ to generic spacetimes?

As is well-known, if M is *globally hyperbolic*, then there are natural generalizations of the *forward/backward propagators*. Namely, there exist unique distributions $G^\vee$ and $G^\wedge$ such that

$$(-\Box + m^2)\zeta^{\vee/\wedge} = f,$$
$$\operatorname{supp}\zeta^{\vee/\wedge} \subset \text{future/past shadow of} \operatorname{supp} f$$

is uniquely solved by

$$\zeta^{\vee/\wedge}(x) := \int G^{\vee/\wedge}(x,y) f(y) |g|^{\frac{1}{2}}(y)\,\mathrm{d}y.$$

A natural generalization of the *Feynman/antiFeynman propagators* is also possible, but less known and more exotic.

Note that $-\Box$ is obviously *Hermitian* (symmetric) on $C_{\mathrm{c}}^{\infty}(M)$ in the sense of the Hilbert space $L^2(M, |g|^{\frac{1}{2}})$. Assume it is *essentially self-adjoint*. Then its resolvent $(-\Box + m^2)^{-1}$ is well defined for complex m^2. For real m^2 we set

$$G^{\mathrm{F}} := \lim_{\epsilon \searrow 0} \frac{1}{-\Box + m^2 - \imath\epsilon}, \quad G^{\overline{\mathrm{F}}} := \lim_{\epsilon \searrow 0} \frac{1}{-\Box + m^2 + \imath\epsilon}.$$

$G^{\mathrm{F}}(x, y)$ and $G^{\overline{\mathrm{F}}}(x, y)$ are the corresponding integral kernels.

Let us describe two arguments in favor of this definition.

If M is *asymptotically stationary and stable in the future and past* then, at least heuristically,

$$-\mathrm{i}G^{\mathrm{F}}(x,y) = \frac{\big(\Omega_+|\mathrm{T}\big(\hat{\psi}(x)\hat{\psi}^*(y)\big)\Omega_-\big)}{\big(\Omega_+|\Omega_-\big)},$$

$$\mathrm{i}G^{\overline{\mathrm{F}}}(x,y) = \frac{\big(\Omega_-|\overline{\mathrm{T}}\big(\hat{\psi}(x)\hat{\psi}^*(y)\big)\Omega_+\big)}{\big(\Omega_-|\Omega_+\big)}.$$

where Ω_- and Ω_+ is the *in vacuum*, resp. the *out vacuum*. These formulas can be found in the old literature, essentially as definitions of G^{F}, $G^{\overline{\mathrm{F}}}$. A more systematic justification can be found in a recent paper by D.Siemssen and JD.

If we use the formalism of *path integrals*, then the generating function is formally defined by

$$Z(J) := \frac{\int \mathrm{e}^{\mathrm{i}S(\psi,\psi^*)+\mathrm{i}\psi J^*+\mathrm{i}\psi^* J}\mathcal{D}\psi\mathcal{D}\psi^*}{\int \mathrm{e}^{\mathrm{i}S(\psi,\psi^*)}\mathcal{D}\psi\mathcal{D}\psi^*}.$$

If the action is *quadratic*

$$S(\psi,\psi^*) = \int \big(\partial_\mu\psi^*(x)\partial^\mu\psi(x) + m^2\psi^*(x)\psi(x)\big)\sqrt{|g|}(x)\,\mathrm{d}x,$$

then we can (rigorously!) evaluate the path integral obtaining

$$Z(J) = \exp\Big(\mathrm{i}\int\int J^*(x)G^{\mathrm{F}}(x,y)J(y)\sqrt{|g|}(x)\sqrt{|g|}(y)\,\mathrm{d}x\,\mathrm{d}y\Big).$$

Essential self-adjointness of the d'Alembertian is easy in some special cases:

- *stationary* spacetimes;
- *Friedmann-Lemaitre-Robertson-Walker* (FLRW) spacetimes
- $1{+}0$-*dimensional* spacetimes
- *deSitter* and (the universal covering of) *anti-deSitter spacetime*.

On a class of *asymptotically Minkowskian* spacetimes essential self-adjointness was recently proven by Vasy and Nakamura-Taira. Essential self-adjointness is destroyed by (space-like or time-like) *boundaries*—this can be repaired by imposing by boundary conditions.

Assume now that M is *globally hyperbolic* and $-\Box$ *is essentially self-adjoint*.

We will say that $-\Box + m^2$ is *special* if

$$G^{\mathrm{F}}(x,y) + G^{\overline{\mathrm{F}}}(x,y) = G^{\vee}(x,y) + G^{\wedge}(x,y).$$

Equivalently, it is special if

$$\mathrm{supp}\,\big(G^{\mathrm{F}}(\cdot,y) + G^{\overline{\mathrm{F}}}(\cdot,y)\big) \subset \text{causal shadow of}\{y\}.$$

If the Klein-Gordon equation is special, then the situation is *super-convenient*! There exist good techniques to compute the Feynman and antiFeynman propagators (because they are defined in the framework of operator theory). For instance, on the Minkowski space we obtain

$$G^{\mathrm{F}/\overline{\mathrm{F}}}(m;x,x') = \frac{\pm\mathrm{i}}{(2\pi)^{\frac{d}{2}}}\Big(\frac{m^2}{(x-y)^2 \pm \mathrm{i}0}\Big)^{\frac{d-2}{4}} K_{\frac{d-2}{2}}\big(m\sqrt{(x-y)^2 \pm \mathrm{i}0}\big),$$

In the *special* situation, the forward/backward propagators can be computed from the formula $G^{\vee/\wedge}(x,y) := \theta(\pm x \mp y)\big(G^{\mathrm{F}}(x,y) + G^{\overline{\mathrm{F}}}(x,y)\big)$.

As usual, we then set $G^{\mathrm{PJ}} := G^{\vee} - G^{\wedge}$. More interestingly, we have a natural candidate for the *two-point function of a distinguished state*:

$$(\Omega \,|\, \hat{\psi}(x)\hat{\psi}^*(y)\Omega) = \frac{1}{\mathrm{i}}(G^{\mathrm{F}} - G^{\wedge}) = \frac{1}{\mathrm{i}}(-G^{\overline{\mathrm{F}}} + G^{\vee}),$$

$$(\Omega \,|\, \hat{\psi}^*(x)\hat{\psi}(y)\Omega) = \frac{1}{\mathrm{i}}(-G^{\overline{\mathrm{F}}} + G^{\wedge}) = \frac{1}{\mathrm{i}}(G^{\mathrm{F}} - G^{\vee}).$$

PART III. EXAMPLES OF SPECIAL SPACETIMES.

As we discussed above, the *Minkowski space* is special if $m^2 \geq 0$. But it is not if $m^2 < 0$ (in the *tachionic* case).

Stationary stable Klein-Gordon equations are special. Recall that stability means that the Hamiltonian is positive definite (which if we only have a mass term corresponds to $m^2 \geq 0$). (This is almost obvious if there is no electrostatic potentials, otherwise see JD-D.Siemssen).

Consider a $1+0$ *dimensional spacetime*. In view of further applications, assume that it is perturbed by a time-dependent potential. Thus the Klein-Gordon operator has the form of a *1-dimensional Schrödinger operator*

$$H := -\partial_t^2 + V(t).$$

Then one can show it is special if H is *reflectionless* at the energy m^2.

For instance, the *symmetric Scarf Hamiltonian*

$$-\partial_t^2 - \frac{\alpha^2 - \frac{1}{4}}{\cosh^2 t}$$

is reflectionless at all energies for $\alpha \in \mathbb{Z} + \frac{1}{2}$.

Let us sketch the theory of Green functions of the 1-dimensional Schrödinger operator. Suppose ψ_1, ψ_2 solve

$$(H + k^2)\psi_i(t) = 0, \qquad i = 1, 2.$$

Then their *Wronskian*

$$\mathcal{W}(\psi_1, \psi_2) := \psi_1(t)\psi_2'(t) - \psi_1'(t)\psi_2(t)$$

does not depend on t.

The function

$$G^{\leftrightarrow}(-k^2; t, s) := \frac{1}{\mathcal{W}(\psi_1, \psi_2)}\big(\psi_1(t)\psi_2(s) - \psi_2(t)\psi_1(s)\big)$$

does not depend on the choice of ψ_1, ψ_2 and defines the so-called *canonical bisolution*, the analog of the Pauli Jordan propagator. From $G^{\leftrightarrow}$ we can define the *forward* and *backward Green functions*:

$$G^{\rightarrow}(-k^2; t, s) := G^{\leftrightarrow}(-k^2, t, s)\theta(t - s),$$
$$G^{\leftarrow}(-k^2; t, s) := -G^{\leftrightarrow}(-k^2, t, s)\theta(s - t).$$

For $\operatorname{Re} k > 0$ we define the left and right *Jost solutions* to be the unique solutions of

$$(H + k^2)\psi_\pm(t,k) = 0, \quad \psi_\pm(t,k) \sim \mathrm{e}^{\mp tk}, \quad \pm t \to \infty.$$

We also introduce the *Jost function*

$$\mathcal{W}(k) := \mathcal{W}\big(\psi_+(\cdot,k), \psi_-(\cdot,k)\big).$$

The *resolvent* of H, denoted $G(-k^2) := (H+k^2)^{-1}$ has the integral kernel

$$G(-k^2;t,s) = \frac{1}{\mathcal{W}(k^2)}\big(\theta(t-s)\psi_+(t,k)\psi_-(s,k) - \theta(s-t)\psi_-(t,k)\psi_+(s,k)\big).$$

We say that H is *reflectionless* if there exist functions $T(\pm \mathrm{i}p)$ such that

$$\psi_+(\pm\mathrm{i}p) = T(\pm\mathrm{i}p)\psi_-(\mp\mathrm{i}p).$$

The *deSitter space* is defined as the submanifold of the $d+1$-dimensional Minkowski *ambient space*:

$$\mathrm{dS}^d := \{X \in \mathbb{R}^{d+1} \mid -X_0^2 + X_1^2 + \cdots + X_d^2 = 1\}.$$

One can look for the Feynman propagator by solving the equation

$$(-\Box_x + m^2) G^{\mathrm{F}}_{\mathrm{dS}}(x,y) = \delta(x-y),$$

and requiring that $G^{\mathrm{F}}_{\mathrm{dS}}(x,y) = G(w)$, where $w = X \cdot Y$ is the product of the vectors in the ambient space. We obtain the Gegenbauer equation

$$\Big((1-w^2)\partial_w^2 - dw\partial_w - (\tfrac{d-1}{2})^2 + m^2\Big) G(w) = 0.$$

We demand the singularities of $G^{\mathrm{F}}_{\mathrm{dS}}$ are analogous to those of the Feynman propagator on the Minkowski space.

Assuming $m > \frac{d-1}{2}$ and setting $\nu := \sqrt{m^2 - (\frac{d-1}{2})^2}$ we obtain

$$G_{\mathrm{dS}}^{\mathrm{F}/\overline{\mathrm{F}}}(m;x,x') = \pm\mathrm{i}\frac{\Gamma(\frac{d-1}{2}+\mathrm{i}\nu)\Gamma(\frac{d-1}{2}-\mathrm{i}\nu)}{(4\pi)^{\frac{d}{2}}}\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\nu}\big(-w\pm\mathrm{i}0\big).$$

Above, $\mathbf{S}_{\alpha,\nu}$ is the *Gegenbauer function* regular at 1 and equal $\frac{1}{\Gamma(\alpha+1)}$ there. *It is special!* We can compute forward/backward propagators, and the distinguished two-point function, called the *Euclidean state* (because it is obtained by the Wick rotation from the Euclidean sphere).

Note that the deSitter space is quite pathological—in particular it is not asymptotically stationary, and the Euclidean state is neither the *in state* nor the *out state*.

There is an alternative approach to the deSitter space based on global coordinates

$$X_0 = \sinh t, \quad X_i = \cosh t \hat{x}_i, \quad \hat{x} \in \mathbb{S}^{d-1}$$

yielding the metric $-\mathrm{d}t^2 + \cosh^2 t\, \mathrm{d}\Omega^2$. This has an FLRW form and yields the Schrödinger operator

$$-\partial_t^2 - \frac{\left(\frac{d-2}{2}\right)^2 - \frac{1}{4} - \Delta_{\mathbb{S}^{d-1}}}{\cosh^2 t} + \left(\frac{d-1}{2}\right)^2.$$

The spectrum of $-\Delta_{\mathbb{S}^{d-1}}$ is $\{l(l+d-2) \ : \ l = 0, 1, 2, \dots\}$, hence we obtain the symmetric Scarf potential with $\alpha = \frac{d-2}{2} + l$. Thus all modes are reflectionless iff d is odd. Consequently, *all modes are special iff d is odd, and they are not if d is even*.

Thus there seems to be a discrepancy with the global approach! However, these two approaches are quite different.

The *Anti-deSitter space* is defined as

$$\mathrm{AdS}^d := \{(X,Y)\in\mathbb{R}^2\times\mathbb{R}^{d-1}\ :\ -X_1^2-X_2^2+Y_1^2+\cdots+Y_{d-1}^2=-1\}.$$

It is stationary, however has timelike loops. By taking the *universal covering* we remove timelike loops. Unfortunately, it is still is not globally hyperbolic: it has trajectories that *escape to infinity in finite time*. To understand its wave propagation we introduce the coordinates

$$X_1=\frac{\cos t}{\cos\rho},\quad X_2=\frac{\sin t}{\cos\rho},\quad X_i=\tan\rho\hat{x}_i;$$

with the metric $\quad \dfrac{1}{\cos^2\rho}(-\,\mathrm{d}t^2+\mathrm{d}\rho^2+\sin^2\rho\,\mathrm{d}\Omega^2).$

Now the Klein-Gordon operator becomes

$$\begin{aligned}&(\tan\rho)^{\frac{d-2}{2}}(\Delta-m^2)(\tan\rho)^{-\frac{d-2}{2}}\\ &=\cos^2\rho\Big(-\partial_t^2+\partial_\rho^2-\frac{\left(\frac{d-3}{2}\right)^2-\frac14-\Delta_{\mathbb{S}^{d-2}}}{\sin^2\rho}-\frac{\left(\frac{d-1}{2}\right)^2-\frac14+m^2}{\cos^2\rho}\Big).\end{aligned}$$

Thus the spatial part of the d'Alembertian is given by the *trigonometric Pöschl-Teller Hamiltonian*

$$H := -\partial_\rho^2 + \frac{\alpha^2 - \frac{1}{4}}{\sin^2 \rho} + \frac{\beta^2 - \frac{1}{4}}{\cos^2 \rho}.$$

This Hamiltonian is essentially self-adjoint if $\alpha^2 \geq 1$ and $\beta^2 \geq 1$, and has a positive Friedrichs extension if $\alpha^2 \geq 0$ and $\beta^2 \geq 0$. Thus, unless $m^2 < -(\frac{d-1}{2})^2$, by taking the Friedrichs extension we obtain a well defined dynamics, and we can define the forward and backward propagators.

The d'Alembertian is essentially self-adjoint. Again, to find the Feynman propagator we set $w := X \cdot X'$ and solve the Gegenbauer equation obtaining

$$G_{\mathrm{AdS}}^{\mathrm{F}/\overline{\mathrm{F}}}(m; x, x') = \pm \mathrm{i} \frac{\sqrt{\pi}\Gamma(\frac{d-1}{2} + \nu)}{\sqrt{2}(2\pi)^{\frac{d}{2}} 2^\nu} \mathbf{Z}_{\frac{d}{2}-1,\nu}\big(-w \pm \mathrm{i}0\big),$$

where $\mathbf{Z}_{\alpha,\lambda}$ is the *Gegenbauer function* behaving as $\frac{w^{-\frac{1}{2}-\alpha-\lambda}}{\Gamma(\lambda+1)}$ at $w \to +\infty$.

Thus properly interpreted *Anti-deSitter space is also special!*

Thank you for your attention