Quantum energy inequality in the Sine-Gordon model

Markus B. Fröb

Institut für Theoretische Physik, Universität Leipzig, Germany

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The Sine–Gordon model

- Two-dimensional QFT with classical action
  $S = -\int \left[\frac{1}{2}\partial^\mu\phi\partial_\mu\phi + \frac{1}{2}m^2\phi^2 - 2g\cos(\beta\phi)\right] \mathrm{d}^2x$
- Different regimes: “finite” for $\beta^2 < 4\pi$, super-renormalisable for $4\pi \le \beta^2 < 8\pi$, just renormalisable for $\beta^2 = 8\pi$, probably non-existing for $\beta^2 > 8\pi$
- In detail: after normal-ordering the interaction $\cos(\beta\phi)$, there are n UV-divergent 1PI diagrams in the range $1 - 1/n \le \beta^2/(8\pi) < n/(n+1)$, i.e. thresholds at 4π, $16/3\pi$, 6π, $32/5\pi$, $20/3\pi$, ...
- Proofs of existence of (renormalised) Euclidean partition function in finite volume for $\beta^2 < 4\pi$ (Fröhlich), for $\beta^2 < 16/3\pi$ (Dimock), for $\beta^2 < 6\pi$ (Bauerschmidt), for $\beta^2 \gtrsim 6\pi$ (Benfatto/Gallavotti/Nicolò), for $\beta^2 < 32/5\pi$ (Nicolò), for $\beta^2 < 8\pi$ (Nicolò/Renn/Steinmann, Dimock/Hurd)
- Proofs of existence of correlation functions of $\mathcal{N}\left(\mathrm{e}^{\pm\mathrm{i}\beta\phi}\right)$ and $\partial_\mu\phi$ in infinite volume for $\beta^2 < 4\pi$ (Fröhlich, Park), for $\beta^2 < 16/3\pi$ (Dimock), correlation functions of ϕ^k and $\mathcal{N}\left(\phi^k\mathrm{e}^{\pm\mathrm{i}\beta\phi}\right)$ in infinite volume for $\beta^2 < 4\pi$ (Fröhlich/Seiler)

- Basic finding: perturbation series in g is convergent!
- Equivalence of Sine–Gordon model with Thirring model with action $\int \left[\mathrm{i}\bar{\psi}\gamma^\mu\partial_\mu\psi - M\bar{\psi}\psi - \lambda j^\mu j_\mu \right] \mathrm{d}^2x$, where $j^\mu = \bar{\psi}\gamma^\mu\psi$ under identification $4\pi/\beta^2 = 1 + \lambda/\pi$, $m = 0$, $-\beta^2/(2\pi)\epsilon^{\mu\nu}\partial_\nu\phi \leftrightarrow j^\mu$, $2g\cos(\beta\phi) \leftrightarrow -M\bar{\psi}\psi$ (Coleman heuristically)
- Proof of equivalence for $\beta^2 = 4\pi$ (Dimock, Bauerschmidt)
- Equivalence of Sine–Gordon model with QED_2 proven for $\beta^2 < 4\pi$ (Fröhlich/Seiler)
- Recent analogous results in Lorentzian signature: convergence of S-matrix and existence of $\partial_\mu\phi$ for $\beta^2 < 4\pi$ (Bahns/Rejzner), existence of $\mathrm{e}^{\pm\mathrm{i}\beta\phi}$ for $\beta^2 < 4\pi$ (Bahns/Fredenhagen/Rejzner)
- Stochastic quantisation for $\beta^2 < 16/3\pi$ (Hairer/Shen), $\beta^2 < 8\pi$ (Chandra/Hairer/Shen)
- Conjectures for other local composite operators such as the stress tensor $T_{\mu\nu} = \partial_\mu\phi\partial_\nu\phi - \frac{1}{2}\eta_{\mu\nu}\partial^\rho\phi\partial_\rho\phi + 2g\eta_{\mu\nu}\cos(\beta\phi)$ in the form factor programme, but convergence of the form factor series unclear/difficult to prove

Our results

- Consider Lorentzian case, $\beta^2 < 4\pi$, quasi-free Hadamard state $\omega^{\Lambda,\epsilon}$ in the vacuum sector, i.e., two-point function $\omega^{\Lambda,\epsilon}(\phi(x)\phi(y))$ differs from the vacuum one by a smooth symmetric bisolution $W(x,y)$ of the (massless) Klein–Gordon equation
- IR cutoff Λ and UV cutoff ϵ, and operators $\mathcal{O}_{\mu\nu} = \partial_\mu\phi\,\partial_\nu\phi$, vertex operators $V_\alpha = \mathrm{e}^{\mathrm{i}\alpha\phi}$
- **Theorem 4**: There exist local counterterms (diverging logarithmically as $\epsilon \to 0$) such that $\lim_{\Lambda,\epsilon\to 0} \omega^{\Lambda,\epsilon}\left(\mathcal{T}\left[\mathcal{O}_{\mu\nu}(z) \otimes \bigotimes_{j=1}^{n} V_{\sigma_j\beta}(x_j)\right]\right)$ exists as a distribution, where $\mathcal{T}$ denotes the time-ordered product. The expectation vanishes unless $\sum_{j=1}^{n}\sigma_j = 0$ (neutrality condition).
- For the stress tensor $T_{\mu\nu} = \mathcal{O}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{O}_\rho{}^\rho + g\eta_{\mu\nu}\left(V_\beta + V_{-\beta}\right)$, the limit exists without counterterms.

- **Theorem 5**: The interacting Gell-Mann–Low expectation values $\omega_{\text{int}}\left(\mathcal{O}_{\mu\nu}(f)\right)$ and $\omega_{\text{int}}\left(T_{\mu\nu}(f)\right)$ are convergent for all $g, f \in \mathcal{S}(\mathbb{R}^2)$ if $\|g\|_\infty$ is small enough, $W(x,y)$ and its first and second derivatives grow at most polynomially, and $\sum_{i,j=1}^{n}[W(x_i,x_j) - W(y_i,x_j) - W(x_i,y_j) + W(y_i,y_j)] \geq 0$ for any x_i and y_i and any $n \in \mathbb{N}$. The second condition holds for example for any W that is the Fourier transform of a positive measure, e.g., a state whose two-point function is thermal in a large range of energies $\left|p^0\right| \in [E_0, E_1]$.
- Gell-Mann–Low formula: $\omega_{\text{int}}(A) = \dfrac{\omega\left(\mathcal{T}\left[A \otimes \sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} S_{\text{int}}^{\otimes n}\right]\right)}{\omega\left(\mathcal{T}\left[\sum_{n=0}^{\infty} \frac{\mathrm{i}^n}{n!} S_{\text{int}}^{\otimes n}\right]\right)}$
- Bound on the series has the form $C \sum_{n=0}^{\infty} n^2 K^n (n!)^{\frac{\beta^2}{4\pi}-1} < \infty$, with C and K depending on f, g, β, W.

- **Theorem 6**: There exists a (finite) redefinition of time-ordered products $\mathcal{T}[\mathcal{O}_{\mu\nu}(x) \otimes V_\alpha(y)] \to \mathcal{T}[\mathcal{O}_{\mu\nu}(x) \otimes V_\alpha(y)] + \delta\mathcal{T}[\mathcal{O}_{\mu\nu}(x) \otimes V_\alpha(y)]$ with $\delta\mathcal{T}$ a local term proportional to $\delta^2(x-y)\mathcal{T}[V_\alpha(y)]$ such that a modified stress tensor is conserved: $\omega_{\text{int}}\left(\hat{T}_{\mu\nu}(\partial^\mu f)\right) = 0$ for all f such that g is constant on the support of f.
- Required modification is one-loop exact: $\hat{T}_{\mu\nu} = T_{\mu\nu} - 2g\frac{\hbar\beta^2}{8\pi}\eta_{\mu\nu}\cos(\beta\phi) = \mathcal{O}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\mathcal{O}_\rho{}^\rho + 2g\left(1 - \frac{\hbar\beta^2}{8\pi}\right)\eta_{\mu\nu}\cos(\beta\phi)$, where we have reinstated $\hbar$ for clarity
- Proves the conjecture made in the form factor programme (Smirnov, Babujian/Karowski)
- The modified stress tensor $\hat{T}_{\mu\nu}$ is unique.
- Existence of higher (conserved) currents has been shown by Zanello (PhD with Bahns).

Quantum energy inequality

- Consider observer following worldline $z^\mu(\tau)$ with four-velocity $v^\mu \equiv \dot{z}^\mu$ satisfying $v^0 \pm v^1 > 0$ (timelike wordline) and $v^\mu v_\mu = -1$ (normalisation)
- Energy density that is seen by the observer: $\epsilon(z) \equiv T_{\mu\nu}(z) v^\mu v^\nu = \frac{1}{2}(v^\mu \partial_\mu \phi)^2 + \frac{1}{2}(w^\mu \partial_\mu \phi)^2 - 2g\cos(\beta\phi)$ with $w_\mu \equiv -\epsilon_{\mu\nu} v^\nu$ spacelike
- Classically, $\epsilon \geq -2g$ is bounded everywhere from below
- In quantum theory, there exist states such that expectation of $\epsilon(z_0)$ is arbitrarily negative at any single point z_0 (Epstein/Glaser/Jaffe, Fewster)
- Nevertheless, smeared energy density is bounded from below = Quantum Energy Inequality (Ford, Fewster, ...)
- Generic form: $E_\omega \equiv \int \omega\,(\epsilon[z(\tau)])\, f^2(\tau)\,\mathrm{d}\tau \geq -K_\omega(f)$ with smearing function f, sufficiently regular state ω
- QEI is state-independent or absolute if K_ω does not depend on ω within class considered (e.g., Hadamard states)

- QEIs are known for free quantum fields in flat and curved spacetimes (scalar, Dirac fermion, vector, Rarita–Schwinger fields)
- Analogous quantum inequalities for observables in OPE of classically positive quantities (Bostelmann/Fewster)
- QEIs for interacting models are more difficult, known only for theories with large symmetry group (2D CFTs Fewster/Hollands) or integrable models (Ising model, Sinh-Gordon/Bullough–Dodd/Federbush/O(N) NLSM at one-particle level Bostelmann/Cadamuro/{Fewster,Mandrysch})
- Analogous inequality for null smearings: Averaged Null Energy Condition, obtained formally from QEI as v^μ becomes null and $f \to 1$

Our results

- Same (Lorentzian) setting as before: $\beta^2 < 4\pi$, quasi-free Hadamard state $\omega^{\Lambda,\epsilon}$ in the vacuum sector, modified stress tensor $\hat{T}_{\mu\nu}$
- Slightly more general condition on state-dependent part $W(x, y)$: $W(f, f^*) = \iint W(x, y) f(x) f^*(y)\, \mathrm{d}^2x\, \mathrm{d}^2y \geq 0$ for (complex) $f \in \mathcal{S}(\mathbb{R}^2)$ with vanishing mean $\int f(x)\, \mathrm{d}^2x = 0$ (conditional positive semidefiniteness)
- **Theorem 7**: For all $f, g \in \mathcal{S}(\mathbb{R}^2)$ the interacting modified stress tensor defined by the Bogoliubov formula $\mathcal{T}_{\text{int}}\left[\hat{T}_{\mu\nu}(z)\right] \equiv \mathcal{T}\left[\mathrm{e}_{\otimes}^{\mathrm{i}S_{\text{int}}}\right]^{\star(-1)} \star \mathcal{T}\left[\hat{T}_{\mu\nu}(z) \otimes \mathrm{e}_{\otimes}^{\mathrm{i}S_{\text{int}}}\right]$ has a finite expectation value in the physical limit: $\lim_{\Lambda,\epsilon\to 0} \omega^{\Lambda,\epsilon}\left(\mathcal{T}_{\text{int}}\left[\hat{T}_{\mu\nu}(f)\right]\right) < \infty$.
  Conservation holds: $\omega^{0,0}\left(\mathcal{T}_{\text{int}}\left[\hat{T}_{\mu\nu}(\partial^\mu f)\right]\right) = 0$ for all f such that g is constant on the support of f
- **Theorem 8**: The same holds with $F^{\mu\nu} = v^\mu(\tau) v^\nu(\tau) f^2(\tau)$ for timelike $v^\mu = \dot{z}^\mu$ and $f \in \mathcal{S}(\mathbb{R})$ instead of f (smearing along the observer's worldline)

- **Theorem 9**: An absolute QEI holds: $E_\omega(f) \equiv \omega^{0,0}\left(\mathcal{T}_{\text{int}}\left[\hat{T}_{\mu\nu}(F^{\mu\nu})\right]\right) \geq -K(z, f, \beta, g)$
- Three contributions to the bound: $K = K_0 + K_V + K_H$
- Free-theory contribution: $K_0 = \frac{1}{24\pi}\int\left[6\left[f'(\tau)\right]^2 + f^2(\tau)\frac{[\ddot{z}^1(\tau)]^2}{1+[\dot{z}^1(\tau)]^2}\right]\mathrm{d}\tau$
- First part arises for straight trajectory $z^\mu(\tau) = \delta^\mu_0\tau$ (Eveson/Fewster)
- Second part comes from difference between straight and actual trajectory, depends on acceleration $\ddot{z}^\mu$
- K_V is contribution from vertex operators, K_H from derivatives of Hadamard parametrices arising from $\mathcal{O}_{\mu\nu}$ terms in stress tensor
- K_0 and K_V stay finite in lightlike limit $\dot{z}^1 \to \infty$, but divergent bounds for K_H in this limit $\Rightarrow$ need different bounds for null energy condition
- Bound not optimal: already for K_0, optimal bound is smaller by factor 3/2 (Flanagan)
- For sufficiently small g (depending also on trajectory z^μ), free-theory contribution K_0 is dominant over K_V and K_H

Details of the proofs

- Step 0: In Lorentzian signature perturbative AQFT framework
- Step 1: Express the expectation value that includes the composite operator $\mathcal{O}_{\mu\nu}$ and vertex operators $V_{\pm\beta}$ by an expectation value of only the vertex operators and (derivatives of) the covariance/two-point function
- Step 2: Identify divergent terms and renormalise
- Step 3: Use the Cauchy determinant representation for the expectation value of the vertex operators (Fröhlich from Deutsch/Lavaud)
- Step 4: Obtain bounds using Hölder and Young inequalities with suitably chosen exponents, taking care to not destroy convergence of the perturbative series

- Step 1: required formula is $\omega^{\Lambda,\epsilon}\left(\mathcal{T}\left[\bigotimes_{j=1}^{n} V_{\alpha_j}(x_j) \otimes (\partial_\mu \phi\, \partial_\nu \phi)(z)\right]\right) = \left[\lim_{z' \to z} \partial_\mu^z \partial_\nu^{z'} W(z, z') + \sum_{i,j=1}^{n} \alpha_i \alpha_j \partial_\mu G^{\mathrm{F}}(z, x_i) \partial_\nu G^{\mathrm{F}}(z, x_j)\right] \omega^{\Lambda,\epsilon}\left(\mathcal{T}\left[\bigotimes_{j=1}^{n} V_{\alpha_j}(x_j)\right]\right)$, where G^{F} is the Feynman propagator (with cutoffs) in the state $\omega^{\Lambda,\epsilon}$
- Proof by induction using causal factorisation of the time-ordered products and symmetry of the Feynman propagator
- Time-ordered products include normal-ordering (with respect to Hadamard parametrix)

- Step 2: divergent terms come from the same double sum with Hadamard parametrix $H_\mu(x_k, z)H_\nu(x_k, z)$, $H_\mu(x, y) \equiv -4\pi \mathrm{i}\partial_\mu H^{\mathrm{F}}(x, y)$
- Use light cone coordinates: $u = (x - y)^0 - (x - y)^1$, $v = (x - y)^0 + (x - y)^1$ such that $(x - y)^2 = -uv$ and obtain $H_u(x, y)H_u(x, y) = 4\pi \mathrm{i}\partial_u^2 H^{\mathrm{F}}(x, y) + \mathrm{i}\pi\delta^2(x - y) + \mathcal{O}(\epsilon)$, $H_v(x, y)H_v(x, y) = 4\pi \mathrm{i}\partial_v^2 H^{\mathrm{F}}(x, y) + \mathrm{i}\pi\delta^2(x - y) + \mathcal{O}(\epsilon)$, $H_u(x, y)H_v(x, y) = -8\pi^2\partial_u\partial_v \left[H^{\mathrm{F}}(x, y)\right]^2 - 2\pi \mathrm{i} \ln(2\mu\epsilon)\delta^2(x - y) + \mathcal{O}(\epsilon)$

- Step 3:
  $\lim_{\Lambda,\epsilon\to 0} \omega^{\Lambda,\epsilon}\left(\mathcal{T}\left[\bigotimes_{j=1}^{n} V_\beta(x_j)\otimes V_{-\beta}(y_j)\right]\right) = \mu^{-n\frac{\beta^2}{2\pi}}\left[\frac{\prod_{1\le j<k\le n}(x_j-x_k)^2(y_j-y_k)^2}{\prod_{j,k=1}^{n}(x_j-y_k)^2}\right]^{\frac{\beta^2}{4\pi}}$ with Feynman iϵ prescription
- For $\beta^2 < 4\pi$, absolute value is integrable by scaling dimension (can ignore iϵ)
- Use light cone coordinates and Cauchy determinant formula to obtain
  $$\left|\left[\frac{\prod_{1\le j<k\le n}(x_j-x_k)^2(y_j-y_k)^2}{\prod_{j,k=1}^{n}(x_j-y_k)^2}\right]^{\frac{\beta^2}{4\pi}}\right| = \left|\det\left(\frac{1}{u_{x_i,y_j}}\right)_{i,j=1}^{n}\right|^{\frac{\beta^2}{4\pi}}\left|\det\left(\frac{1}{v_{x_i,y_j}}\right)_{i,j=1}^{n}\right|^{\frac{\beta^2}{4\pi}}$$
- Bound on Cauchy determinant: $\left|\det\left(\frac{1}{u_{x_i,y_j}}\right)_{i,j=1}^{n}\right| \le \sum_\pi \prod_{j=1}^{n}\left|\frac{1}{u_{x_j,y_{\pi(j)}}}\right|$
- Step 4: Further bounds using Hölder and Young inequalities (long and messy)

- No essential changes when using Bogoliubov formula instead of Gell-Mann–Low for space-time smearing (time-ordering almost irrelevant for $\beta^2 < 4\pi$)
- For smearing along one-dimensional worldline, small modifications are needed
- Essential ingredient: change of variables from τ to $u(z(\tau))$ or $v(z(\tau))$, using that $\partial_\tau \{u, v\}(z(\tau)) = \sqrt{1 + (\dot{z}^1)^2} \pm \dot{z}^1 > 0$ for time-like worldline
- Proof of QEI: As in free theory, point-splitting of energy density and separation into a manifestly non-negative and a bounded part
- Candidate for non-negative part: $\frac{1}{2} f(\tau) f(\tau') \Big[v^\mu(\tau) v^\nu(\tau') + w^\mu(\tau) w^\nu(\tau') \Big] \omega^{0,0} \left(\left(\mathcal{T}_{\text{int}} \left[\partial_\mu \phi(z) \right] \right)^\dagger \star \mathcal{T}_{\text{int}} \left[\partial_\nu \phi(z') \right] \right)$
- Issue: $\omega^{0,0}$ is only conditionally positive because of state-dependent part W, which is positive semi-definite only for smearings with vanishing mean
- Solution: construct positive $\tilde{\omega}^{\Lambda,\epsilon}$ such that $\lim_{\Lambda,\epsilon\to 0} \left[\tilde{\omega}^{\Lambda,\epsilon}(A) - \omega^{\Lambda,\epsilon}(A) \right] = 0$ for all A with $\lim_{\Lambda,\epsilon\to 0} \omega^{\Lambda,\epsilon}(A) < \infty$

- Essentially, $\tilde{\omega}^{\Lambda,\epsilon}$ is the massive extension of $\omega^{\Lambda,\epsilon}$ with mass $\sim \Lambda$
- Construction somewhat subtle and specific to 2D, uses conditionally positive semidefiniteness of W
- Remainder of energy density is bounded: free-theory part combines methods from Fewster/Smith (straight trajectory) and Flanagan (deviation), interaction part uses bounds from Theorem 8 (smearing of stress tensor)
- Essential ingredient for bounds of interaction part is again the change of variables from τ to $u(z(\tau))$ or $v(z(\tau))$, valid for time-like worldline
- State dependence enters bounded part only as $\exp\left[-\sum_{i,j=1}^{n}\left[W(x_i,x_j)-W(x_i,y_j)-W(y_i,x_j)+W(y_i,y_j)\right]\right] \leq 1$ because of conditionally positive semidefiniteness of W
- State dependence of positive part is in addition through $v^\mu v^\nu \partial_\mu^x \partial_\nu^y W(x,y)\big|_{x=y=z(\tau)}$, which can be arbitrarily large

Thank you for your attention

Questions?

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