Point potentials on the Euclidean space, hyperbolic space and sphere in any dimension

(joint work with J. Dereziński and B. Ruba)

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Consider the Euclidean space $\mathbb{R}^d$ and the hyperbolic space and sphere in dimension d:

$$\mathbb{H}^d := \{x \in \mathbb{R}^{1+d} \mid [x|x] := x_0^2 - \vec{x}^2 = 1\}, \qquad \mathbb{S}^d := \{x \in \mathbb{R}^{1+d} \mid (x|x) := x_0^2 + \vec{x}^2 = 1\},$$

and the operators

$$H_d := -\Delta_d, \qquad H_d^h := -\Delta_d^h - \left(\tfrac{d-1}{2}\right)^2, \qquad H_d^s := -\Delta_d^s + \left(\tfrac{d-1}{2}\right)^2,$$

with the Laplace-Beltrami operators Δ_d, Δ_d^h, Δ_d^s on $\mathbb{R}^d$, $\mathbb{H}^d$, $\mathbb{S}^d$. Their spectra are

$$\sigma(H_d) = \sigma(H_d^h) = [0, \infty[, \qquad \sigma(H_d^s) = \left\{\left(l + \tfrac{d-1}{2}\right)^2 \mid l = 0, 1, \dots\right\} \subset [0, \infty[.$$

We want to describe *point-like perturbations* of $H_d^\bullet + \beta^2$, $\mathrm{Re}(\beta) > 0$, in *any* dimension.

In dimensions $d \geq 4$, $\nexists$ point-like perturbations because Δ_d is ess. s.a. on $C_c^\infty(\mathbb{R}^d \setminus 0)$.

After a *renormalization*, one can define objects that can be interpreted as Green functions in some sense.

Point-like perturbations

The generalized integral

Green functions in arbitrary dimensions

Green functions of the unperturbed operators

Let $\mathrm{Re}(\beta) > 0$ and denote the integral kernels of $\left(H_d^\bullet + \beta^2\right)^{-1}$ by $G_d^\bullet(-\beta^2; x, x')$. Then

$$G_d(-\beta^2; x, x') = (2\pi)^{-\frac{d}{2}}\left(\tfrac{\beta}{|x-x'|}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}\big(\beta|x-x'|\big),$$

$$G_d^h(-\beta^2; x, x') = \frac{\sqrt{\pi}\,\Gamma\left(\frac{d-1}{2}+\beta\right)}{\sqrt{2}(2\pi)^{\frac{d}{2}}2^\beta}\mathbf{Z}_{\frac{d}{2}-1,\beta}\big([x|x']\big),$$

$$G_d^s(-\beta^2; x, x') = \frac{\Gamma\left(\frac{d-1}{2}+\mathrm{i}\beta\right)\Gamma\left(\frac{d-1}{2}-\mathrm{i}\beta\right)}{(4\pi)^{\frac{d}{2}}}\mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\beta}\big(-(x|x')\big).$$

$K_\nu(z)$ is the *MacDonald function* (or modified Bessel function of the 2nd kind).
$\mathbf{Z}_{\alpha,\lambda}(z)$ and $\mathbf{S}_{\alpha,\lambda}(z)$ are *Gegenbauer functions*:

$$\mathbf{Z}_{\alpha,\lambda}(z) = \frac{(z\pm 1)^{-\frac{1}{2}-\alpha-\lambda}}{\Gamma(1+\lambda)}\;{}_2F_1\Big(\frac{1}{2}+\lambda, \frac{1}{2}+\lambda+\alpha; 1+2\lambda; \frac{2}{1\pm z}\Big),$$

$$\mathbf{S}_{\alpha,\lambda}(z) = \frac{1}{\Gamma(1+\alpha)}\;{}_2F_1\Big(\frac{1}{2}+\alpha+\lambda, \frac{1}{2}+\alpha-\lambda; \alpha+1; \frac{1-z}{2}\Big).$$

Point potentials

Suppose that H_d^γ is a self-adjoint extension of H_d restricted to $C_c^\infty(\mathbb{R}^d \setminus 0)$.
By the resolvent identities, the Green function of H_d^γ should satisfy

$$\begin{aligned}(-\Delta_x - z)G_d^\gamma(z,x,x') &= \delta(x-x'), \quad x \neq 0,\\ G_d^\gamma(z;x,x') &= G_d^\gamma(z;x',x),\\ \partial_z G_d^\gamma(z;x,x') &= -\int G_d^\gamma(z;x,y)G_d^\gamma(z;y,x')\mathrm{d}y.\end{aligned}$$

This is solved by a *Krein type resolvent*

$$G_d^\gamma(z;x,x') = G_d(z;x,x') + \frac{G_d(z;x,0)G_d(z;0,x')}{\gamma + \Sigma_d(z)},$$

where the *self-energy* $\Sigma_d(z)$ is defined up to an integration constant (if it *is* defined):

$$\partial_z \Sigma_d(z) = -\sigma_d(z) = -\int_{\mathbb{R}^d} G_d(z;0,y)^2 \mathrm{d}y.$$

The integral $\sigma_d(z)$ converges only in dimensions $d = 1, 2, 3$.

It diverges if $d \geq 4$, reflecting the fact that Δ_d is ess. s.a. on $C_c^\infty(\mathbb{R}^d \setminus 0)$ for $d \geq 4$.

We will construct renormalized Green functions by giving a meaning to $\sigma_d(z)$ in any dimension.

There will occur *anomalies*. These introduce an ambiguity in the construction, corresponding to a *renormalization freedom*.

(The curved cases are analogous.)

Definition

Let $a \in \mathbb{R}$. A function f on $]a,\infty[$ is *integrable in the generalized sense* if it is integrable on $]a+1,\infty[$ and there exists a finite set $\Omega \subset \mathbb{C}$ and complex coefficients $(f_k)_{k\in\Omega}$ s.t.

$$f - \sum_{k\in\Omega} f_k (r-a)^k$$

is integrable on $]a, a+1[$. We define

$$\mathrm{gen}\int_a^\infty f(r)\mathrm{d}r := \sum_{k\in\Omega\setminus\{-1\}} \frac{f_k}{k+1} + \int_a^{a+1} \Big(f(r) - \sum_{k\in\Omega} f_k (r-a)^k \Big)\mathrm{d}r + \int_{a+1}^\infty f(r)\mathrm{d}r.$$

The generalized integral is a *linear continuation* of the standard integral.
(It reduces to the latter if f is integrable.)

The idea of the generalized integral goes back to Hadamard and Riesz. It is ...

- ... related to the *extension of homogeneous distributions* and the *barred integral* of Lesch.
- ... *translation invariant*:

$$\operatorname{gen}\int_a^\infty f(r)\mathrm{d}r = \operatorname{gen}\int_{a-b}^\infty f(u+b)\mathrm{d}u, \quad b \in \mathbb{R}.$$

- ... invariant with respect to power transformations:

$$\operatorname{gen}\int_0^\infty f(r)\mathrm{d}r = \operatorname{gen}\int_0^\infty f(u^\alpha)\,\alpha u^{\alpha-1}\mathrm{d}u, \quad \alpha > 0.$$

- ... has (in general) a *scaling anomaly*:

$$\operatorname{gen}\int_0^\infty f(\alpha u)\alpha\mathrm{d}u = \operatorname{gen}\int_0^\infty f(r)\mathrm{d}r - f_{-1}\ln\alpha, \quad \alpha > 0.$$

In particular, $\mathrm{gen}\!\int$ is *coordinate dependent*. More general:

$$\mathrm{gen}\!\int_0^\infty f(g(u))g'(u)\mathrm{d}u - \mathrm{gen}\!\int_0^\infty f(r)\mathrm{d}r = -f_{-1}\ln g'(0) + \sum_{l\in(\mathbb{N}+1)\cap\Omega} c_l(g)\, f_{-l},$$

for g a well-behaved coordinate trf. with $g(0)=0$, $g'(0)\neq 0$.

$c_l(g)$ are g-dependent constants, which depend on finitely many $g^{(n)}(0)$.

Definition

The generalized integral is called *anomalous* if there is $k\in\mathbb{N}$ s.t. $f_{-k}\neq 0$.

The generalized integral with a specified coordinate chooses a *distinguished* renormalization.

One often encounters functions that depend on an additional parameter α, which satisfy

$$\mathrm{gen}\int_0^\infty f(r,\alpha)\mathrm{d}r = \sum_{n=0}^{N} \frac{f_n(\alpha)}{\alpha+n+1} + \int_0^1 \Big(f(r,\alpha) - \sum_{n=0}^{N} r^{\alpha+n} f_n(\alpha) \Big) \mathrm{d}r + \int_1^\infty f(r,\alpha)\mathrm{d}r \qquad (1)$$

for $-\alpha \notin \{1,\ldots,N\}$, s.t. the RHS is holomorphic in α away from the poles at $-1,\ldots,-N$.

Non-anomalous case

In the non-anomalous case, (1) can be computed by analytic continuation from the region of α where the generalized integral coincides with the standard integral.

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for $-\alpha \notin \{1,\ldots,N\}$, s.t. the RHS is holomorphic in α away from the poles at $-1,\ldots,-N$.

Anomalous case: dimensional regularization

Let $m \in \{1,\ldots,N\}$. The RHS of (1) has a simple pole at $\alpha = -m$ with residue $f_{m-1}(-m)$.

Its finite part is

$$\mathop{\mathrm{fp}}_{\alpha\to -m} \mathrm{gen}\int_0^\infty f(r,\alpha)\mathrm{d}r = \lim_{\alpha\to -m}\left(\mathrm{gen}\int_0^\infty f(r,\alpha)\mathrm{d}r - \frac{f_{m-1}(-m)}{\alpha+m}\right)$$

and

$$\mathrm{gen}\int_0^\infty f(r,-m)\mathrm{d}r = \mathrm{fp}_{\alpha\to -m}\mathrm{gen}\int f(r,\alpha)\mathrm{d}r - f'_{m-1}(-m).$$

Computation of the self-energies

The self energies $\Sigma_d^{\bullet}(\rho)$ are determined via

$$\partial_\rho \Sigma_d(\rho) = \frac{\rho^{\frac{d-2}{2}} \pi^{\frac{d}{2}}}{(2\pi)^d \Gamma(\frac{d}{2})} \operatorname{gen}\int_0^\infty K_{\frac{d}{2}-1}(\sqrt{\rho} r)^2 \mathrm{d}r^2,$$

$$\partial_\rho \Sigma_d^h(\rho) = \frac{\pi \Gamma\left(\frac{d-1}{2} + \sqrt{\rho}\right)^2}{2^{2\sqrt{\rho}+1} (4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} \operatorname{gen}\int_1^\infty \mathbf{Z}_{\frac{d}{2}-1,\sqrt{\rho}}(w)^2 (w^2-1)^{\frac{d}{2}-1} \mathrm{d}2w,$$

$$\partial_\rho \Sigma_d^s(\rho) = \frac{\Gamma\left(\frac{d-1}{2} + \mathrm{i}\sqrt{\rho}\right)^2 \Gamma\left(\frac{d-1}{2} - \mathrm{i}\sqrt{\rho}\right)^2}{(4\pi)^{\frac{d}{2}} \Gamma\left(\frac{d}{2}\right)} \operatorname{gen}\int_{-1}^1 \mathbf{S}_{\frac{d}{2}-1,\mathrm{i}\sqrt{\rho}}(w)^2 (1-w^2)^{\frac{d}{2}-1} \mathrm{d}2w.$$

In all three cases:

- For $d = 1, 2, 3$, the generalized integral coincides with the standard integral.
- For *odd* $d \geq 5$, the generalized integral can be computed by analytic continuation.
- For *even* $d \geq 4$, the generalized integral is anomalous.

We consider the flat case (the curved cases are analogous but more complicated). Then

$$\Sigma_d(\beta^2) = \begin{cases} -\frac{1}{2\beta} & d = 1; \\ \frac{\ln(\beta^2)}{4\pi} & d = 2; \\ \frac{\beta}{4\pi} & d = 3. \end{cases}$$

In higher dimension, application of the generalized integral yields

$$\Sigma_d(\beta^2) = \begin{cases} \frac{(-1)^{\frac{d+1}{2}}\beta^{d-2}}{(4\pi)^{\frac{d-1}{2}}2(\frac{1}{2})_{\frac{d-1}{2}}}, & d \text{ odd}; \\ \frac{(-1)^{\frac{d}{2}+1}\beta^{d-2}}{(4\pi)^{\frac{d}{2}}(\frac{d}{2}-1)!}\left(2 - 2\psi\left(\frac{d}{2}\right) + \ln\frac{\beta^2}{4}\right), & d \text{ even}. \end{cases}$$

Regularized Green functions

Recall the the Krein-type formula: $G_d^{\bullet,\gamma}(z;x,x') = G_d^{\bullet}(z;x,x') + \dfrac{G_d^{\bullet}(z;x,0)G_d^{\bullet}(z;0,x')}{\gamma+\Sigma_d^{\bullet}(z)}$.

In the *Euclidean case*, this yields

$$G_d^{\gamma}(-\beta^2;x,x') = \begin{cases} \frac{\mathrm{e}^{-\beta|x-x'|}}{2\beta} + \frac{\mathrm{e}^{-\beta|x|}\mathrm{e}^{-\beta|x'|}}{(2\beta)^2(\gamma-\frac{1}{2\beta})}, & d=1; \\ \frac{K_0(\beta|x-x'|)}{2\pi} + \frac{K_0(\beta|x|)K_0(\beta|x'|)}{(2\pi)^2(\gamma+\frac{\ln\beta^2}{4\pi})}, & d=2; \\ \frac{\mathrm{e}^{-\beta|x-x'|}}{4\pi|x-x'|} + \frac{\mathrm{e}^{-\beta|x|}\mathrm{e}^{-\beta|x'|}}{(4\pi)^2|x||x'|(\gamma+\frac{\beta}{4\pi})}, & d=3, \end{cases}$$

or in any dimension

$$G_d^{\gamma}(-\beta^2;x,x') = \frac{1}{(2\pi)^{\frac{d}{2}}}\left(\frac{\beta}{|x-x'|}\right)^{\frac{d}{2}-1} K_{\frac{d}{2}-1}(\beta|x-x'|)$$
$$+\frac{1}{(2\pi)^d}\left(\frac{\beta^2}{|x||x'|}\right)^{\frac{d}{2}-1}\frac{K_{\frac{d}{2}-1}(\beta|x|)K_{\frac{d}{2}-1}(\beta|x'|)}{\gamma+\Sigma_d(\beta^2)}.$$

The perturbed Green functions on $\mathbb{H}^d$ and $\mathbb{S}^d$ can also be computed explicitly.

Then it is easy to find the Green functions on the respective spaces with curvature $\pm\frac{1}{R^2}$.

The latter *converge* to the flat Green function if R becomes large.

This is due to the asymptotics of Gegenbauer functions,

$$\frac{\pi e^{-\pi\beta}(\sin\theta)^{\alpha+\frac{1}{2}}}{2^\alpha\theta^{\alpha+\frac{1}{2}}}\mathbf{S}_{\alpha,\pm i\beta}(-\cos\theta) = (\theta\beta)^{-\alpha}K_\alpha(\beta\theta)\big(1+O(\beta^{-1})\big);$$
$$\frac{\sqrt{\pi}\Gamma(\frac{1}{2}-\alpha+\lambda)(\sinh\theta)^{\alpha+\frac{1}{2}}}{2^{\lambda+\frac{1}{2}}\theta^{\alpha+\frac{1}{2}}}\mathbf{Z}_{\alpha,\lambda}(\cosh\theta) = (\lambda\theta)^{-\alpha}K_\alpha(\lambda\theta)\big(1+O(\lambda^{-1})\big),$$

and of the respective *generalized integrals*.

It is *non-trivial* that asymptotics of the integrand imply asymptotics of the generalized integral.

Poles of the perturbed Green function on the sphere

The perturbed Green functions have additional poles, caused by the vanishing of $\gamma + \Sigma_d^{\bullet}(\beta^2)$.

On $\mathbb{R}^d$ and $\mathbb{H}^d$ and in $d = 1, 2, 3$, they correspond to *one* new bound state. For $d \geq 4$, the situation is more complicated but not very rich.

On $\mathbb{S}^d$ resp. $\mathbb{S}_R^d$, the situation is more interesting because $-\Delta_d^s$ has *discrete spectrum*.

One finds: $G_{d,R}^{s,\gamma}(z)$ possess a sequence of poles indexed by $l \in \mathbb{N}_0$.

The *leading correction* to the unperturbed eigenvalues $\frac{\omega_{d,l}^2}{R^2}$, $\omega_{d,l} := \frac{d-1}{2} + l$, is always $\sim R^{-d}$.

Interpretation of the Green functions in higher dimensions

Suppose that V is a potential with narrow support.

If $G_d^V(-\beta^2; x, x')$ is the integral kernel of $(\beta^2 - \Delta_d + V)^{-1}$, then $G_d^\gamma(-\beta^2; x, x')$ should approximate $G_d^V(-\beta^2; x, x')$ in a suitable sense.

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Renormalized Green functions as limits of true Green functions

In the flat case, we know that $G_d^\gamma(-\beta^2; x, x')$ can be seen as limit of Green functions of suitably scaled rank-one perturbations $V_\epsilon := |f_\epsilon)(g_\epsilon|$, which act on $\psi \in L^2(\mathbb{R}^d)$ as

$$\Big(|f_\epsilon)(g_\epsilon|\psi\Big)(x) := f_\epsilon(x) \int \overline{g_\epsilon(y)} \psi(y) \mathrm{d}y.$$

There are *renormalization ambiguities* corresponding to subleading polynomials in the energy.

Thank you for your attention!