Non-Fock Ground States in the Translation-Invariant Nelson Model Revisited Non-Perturbatively

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Nelson Model

Given a complex Hilbert space $\mathfrak{h}$ we define the bosonic Fock space over $\mathfrak{h}$ as

$$\mathcal{F}(\mathfrak{h}) := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} \mathfrak{h}^{\otimes_{\mathrm{s}} n},$$

where $\otimes_{\mathrm{s}}$ denotes the symmetric tensor product. In this work we define

$$\mathcal{F} := \mathcal{F}(L^2(\mathbb{R}^d))$$

and use the canonical identification $L^2(\mathbb{R}^d)^{\otimes_{\mathrm{s}} n} = L^2_{\mathrm{sym}}([\mathbb{R}^d]^n)$, where sym stands for the subspace wich is totally symmetric with respect to interchange of any of the n arguments.

For a selfadjoint operator A in $\mathfrak{h}$ we define

$$d\Gamma(A) := 0 \oplus \bigoplus_{n=1}^{\infty} \sum_{j=1}^{n} \mathbf{1}^{\otimes(j-1)} \otimes A \otimes \mathbf{1}^{\otimes(n-j)}$$

Given $\psi = (\psi_n)_{n\in\mathbb{N}_0} \in \mathcal{F}$ we define the annihilation operator

$$(a(k)\psi)_n(k_1, ..., k_n) = \sqrt{n+1}\psi_{n+1}(k_1, ..., k_n, k), \quad k, k_j \in \mathbb{R}^d.$$

$$a(f) := \int d^3k\overline{f(k)}a(k), \quad a^*(f) := [a(f)]^*$$

$$\phi(f) := a(f) + a(f).$$

Hamiltonian

The total momentum (generator of translations) is given by $P_{\rm f} = (P_{{\rm f},1}, ..., P_{{\rm f},d})$ with $P_{{\rm f},j} := d\Gamma(\mathbf{k}_j)$ for $j = 1, ..., d$, where

$$\mathbf{k}_j : \mathbb{R}^d \to \mathbb{R}, \quad (k_1, ..., k_d) \mapsto k_j.$$

The fiber Hamiltonian of the translation invariant Nelson Model is

$$H(\Theta, \Omega, v, P) := \Theta(P - P_{\rm f}) + d\Gamma(\Omega) + \varphi(v), \tag{1}$$

where

- $P \in \mathbb{R}^d$.
- $\Omega : \mathbb{R}^d \to [0, \infty)$, subadditive, strictly positive a.e. and $\Omega(k) \geq |k|$
- $\Theta : \mathbb{R}^d \to \mathbb{R}$ with $\Theta(p) = \Theta_{\rm nr}(p) := \frac{p^2}{2M}$ or $\Theta(p) = \Theta_{\rm sr}(p) := \sqrt{M^2 + p^2}$, $M > 0$.
- $v \in L^2(\mathbb{R}^d)$ such that $|\cdot|^{-1/2} v \in L^2(\mathbb{R}^d)$ (physically interesting case $v(k) = |k|^{-1/2} 1_{|k|<\Lambda}$ and $d = 3$)

Lemma 1

For any $P \in \mathbb{R}^d$ the operator $H(\theta, \Omega, v, P)$ is lower-semibounded and self-adjoint and on the natural domain of $\Theta(P_{\rm f}) + d\Gamma(\Omega)$.

Let $\omega(k) = |k|$. We define

$$H_{\mathrm{nr}}(P) := H(\Theta_{\mathrm{nr}}, \omega, v, P), \quad H_{\mathrm{sr}}(P) := H(\Theta_{\mathrm{sr}}, \omega, v, P)$$

Henceforth, let $\# = \mathrm{nr}$ or $\# = \mathrm{sr}$.

Do these Hamiltonians have a ground state?

- Relevance: Construction of Scattering States [Fröhlich, Pizzo, Dybalski, Graf, Beaud, ...]
- Answer: In general no [Faddeev-Kulish, Bloch-Nordsieck, Fröhlich, Pizzo, Dam, ...]

IR-Renormalized Hamiltonian

Remedy: Consider non-Fock space representations. Define

$$\omega_n(k) := \sqrt{|k|^2 + \mu_n}, \quad \mu_n = 1/n.$$

and for $Q \in \mathbb{R}^d$ with $|Q| < 1$ define

$$f_{Q,n}(k) := \frac{v(k)}{\omega_n(k) - k \cdot Q}, \quad k \in \mathbb{R}^d.$$

Note: $f_{Q,n} \in L^2(\mathbb{R}^d)$ but in general $f_{Q,\infty} \notin L^2(\mathbb{R}^d)$.
Let $W(f) = \exp(i[a(f) - a^*(f)])$.

Theorem 1 (Construction of IR-renormalized Hamiltonian)

Let $P, Q \in \mathbb{R}^d$ with $|Q| < 1$. Then the operators

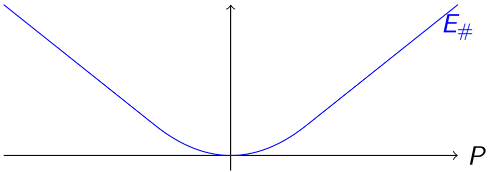
$$W(f_{Q,n}) H_\#(P) W(f_{Q,n})^*$$

converge to a selfadjoint and lower bounded operator $\widehat{H}_\#(P; Q)$ in norm-resolvent sense as $n \to \infty$.

Previous: Fröhlich, Pizzo, Bachmann-Deckert-Pizzo'12, (UV) Nelson'64, Deckert-Pizzo'14, ...

Define

$$E_\#(P) := \inf \sigma(H_\#(P)),$$



$$\mathcal{D}_{\#,k} := \{P \in \mathbb{R}^d : \xi \mapsto E_\#(\xi) \text{ is } k\text{-times differentiable at } \xi = P\}.$$

Set

$$\mathcal{B}_{\mathrm{nr}} := \{P \in \mathbb{R}^d : |P| < M\}, \quad \mathcal{B}_{\mathrm{sr}} := \mathbb{R}^d.$$

Lemma 2 (Convexity)

The set $\mathcal{D}_{\#,k}$ has full Lebesgue measure for $k = 1, 2$. Further, for all $P \in \mathcal{D}_{\#,1} \cap \mathcal{B}_{\#}$, we have $|\nabla E_{\#}(P)| < 1$.

Theorem 3 (Existence of Ground States for IR-renormalized Hamiltonian)

Let $P \in \mathcal{D}_{\#,2} \cap \mathcal{B}_{\#}$. Then $\widehat{H}_{\#}(P, \nabla E_{\#}(P))$ has a ground state.

Previous: Fröhlich, Pizzo, Bachmann-Deckert-Pizzo '12

It was shown by Abdessalam-H 2012 that $E_{\mathrm{nr}}(\cdot)$ is analytic in a neighborhood of P. This implies the following corollary.

Corollary 4

Let $M = 1$ and $|P| < 1$. If $\| |\cdot|^{-1/2} v\| < (1 - |P|)^{2/3}$, then the operator $\widehat{H}_{\mathrm{nr}}(P, \nabla E_{\mathrm{nr}}(P))$ has a ground state.

Proof of Construction of IR-renormalized Hamiltonian

Use elemetry properties of Weyl operators to show the following lemma.

Lemma 5

Let $f \in L^2(\mathbb{R}^d)$ such that $\omega f, \omega^{1/2} f \in L^2(\mathbb{R}^d)$. Then

$$\begin{aligned} & W(f) H_{\#}(P) W(f)^* \\ & = \Theta_{\#}(\mathbf{v}_P(f)) + d\Gamma(\omega) - \varphi(\omega f) + \mathfrak{s}(f, \omega f) + \phi(v) - 2\mathrm{Re}\mathfrak{s}(f, v) \\ & =: T_{\#}(P, f), \end{aligned}$$

where

$$\mathfrak{s}(f, g) := \int \overline{f}(k) g(k) dk$$

$$\mathbf{v}_P(f) := P - P_{\mathrm{f}} + \phi(\mathbf{k} f) - \mathfrak{s}(f, \mathbf{k} f),$$

is selfadjoint on the natural domain of $\Theta_{\#}(P_{\mathrm{f}}) + d\Gamma(\omega)$.

Idea of the proof of the Theorem about the construction of the IR-renormalized Hamiltonian:
In the non-relativistic case use the resolvent identity and a series of estimates

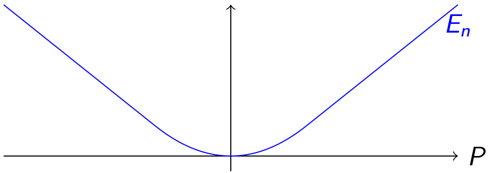
$$\begin{aligned} &\|(T_{\rm nr}(P,f)+i)^{-1}-(T_{\rm nr}(P,g)+i)^{-1}\| \\ &= \|(T_{\rm nr}(P,f)+i)^{-1}(T_{\rm nr}(P,g)-T_{\rm nr}(P,f))(T_{\rm nr}(P,g)+i)^{-1}\| \\ &\leq C(\|\omega f\|+\|\omega^{1/2}f\|)(\|\omega(f-g)\|+\|\omega^{1/2}(f-g)\|)(\|\omega g\|+\|\omega^{1/2}g\|) \end{aligned}$$

R.H.S. tends to zero for $f=f_{n,Q}$ and $g=f_{m,Q}$ and $n,m\to\infty$.
In the semi-relativistic case use in addition the identity

$$F^{1/2}-E^{1/2}=\frac{1}{\pi}\int_0^\infty t^{1/2}(t+F)^{-1}(F-E)(t+E)^{-1}dt.$$

Idea: Add a positive photon mass such that there exists a ground state and follow the ground state as the photon mass tends to zero.
For simplicity we drop # in the notation and assume $M = 1$.
Let

$$H_n(P) := H(\Theta, \omega_n, v, P), \quad E_n(P) := \inf \sigma(H_n(P))$$



Define

$$\mathcal{I}_n := \{P \in \mathbb{R}^d : E_n(P) < \inf_{k \in \mathbb{R}^d} (E_n(P-k) + \omega_n(k))\}$$

$$\Delta_{P,n} := \sup_{k \in \mathbb{R}^d} \frac{\omega_n(k)}{E_n(P-k) - E_n(P) + \omega_n(k)}$$

Theorem 6 (Gross, Fröhlich, Spohn, Pizzo, Dam)

Let e stand for E or E_n.

- $e(\cdot)$ *is continuous, a.e. differentiable, and a.e. twice differentiable.*
- $e(P-k) - e(P) \geq \begin{cases} -2|k||P| & \text{for } |k| \leq |P|, \\ -2|P|^2 & \text{else.} \end{cases}$
- *If* $e(\cdot)$ *is differentiable at* P, *then* $|\nabla e(P)| \leq |P|$.

Corollary 7

nr: If $P \in \mathbb{R}^d$ *and* $|P| < 1$, *then* $P \in \mathcal{I}_n$ *and* $\Delta_{P,n} \leq (1 - |P|)^{-1}$.
sr: If $P \in \mathbb{R}^d$, *then* $P \in \mathcal{I}_n$ *and* $\sup_n \Delta_{P,n} < \infty$.

Theorem 8 (Fröhlich, Derezinski, Gerard, Griesemer, Lieb, Loss, Moeller)

Let $P \in \mathcal{I}_n$, *then* $E_n(P)$ *is a discrete eigenvalue of* $H_n(P)$.

We denote by $\psi_{P,n}$ a normalized ground state of $H_n(P)$.

Let

$$g_{P,n} := f_{\nabla E_n(P),n} = \frac{v(k)}{\omega_n(k) - k \cdot \nabla E_n(P)}.$$

Recall that $g_{P,n} \in L^2(\mathbb{R}^d)$ is well-defined if $|\nabla E_n(P)| < 1$.

Theorem 9 (Fröhlich, Griesemer-Lieb-Loss, Pizzo, Dam, H-Siebert)

- $\lim_{n\to\infty} E_n(P) = E(P)$.
- *If $E(\cdot)$ is differentiable at P, then* $\nabla E(P) = \lim_{n\to\infty} \nabla E_n(P)$.
- *If $E(\cdot)$ is twice differentiable at P, then* $\limsup_{n\to\infty} \max_{i=1,\ldots,d} \partial_i^2 E_n(P) < \infty$.

By a compactness result, Theorem 18 (below), the set $\{W(g_{P,n})\psi_{P,n} : n \in \mathbb{N}\}$ **is relatively compact.**

Thus the limit $\widehat{\psi}_P := \lim_{n\to\infty} W(g_{P,n})\psi_{P,n}$ exists
(by possibly going over to a subsequence).
It remains to show that $\widehat{\psi}_P$ is a ground state of $\widehat{H}(P, \nabla E(P))$:

First note that by Theorem 1 we have

$$\inf \sigma\left(\widehat{H}(P, \nabla E(P))\right) = E(P),$$

and $W(g_{P,n})\psi_{P,n} \in \mathcal{D}(\widehat{H}(P, \nabla E(P)))$.

$$\begin{aligned}
0 &\leq \left\langle W(g_{P,n})\psi_{P,n}, \left(\widehat{H}(P, \nabla E(P)) - E(P)\right) W(g_{P,n})\psi_{P,n} \right\rangle \\
&= \left\langle W(g_{P,n})\psi_{P,n}, \left[\widehat{H}(P, \nabla E(P)) - W(g_{P,n})H(P)W(g_{P,n})^*\right] W(g_{P,n})\psi_{P,n} \right\rangle \\
&\quad + \langle \psi_{P,n}, (H(P) - E(P))\, \psi_{P,n} \rangle \\
&\longrightarrow 0 \quad (n \to \infty),
\end{aligned}$$

where the first term converges to zero by norm resolvent convergence in Theorem 1 and the second term by

$$0 \leq H(P) - E(P) \leq (H_n(P) - E_n(P)) + (E_n(P) - E(P))$$

Using the lower-semicontinuity of closed quadratic forms, we find

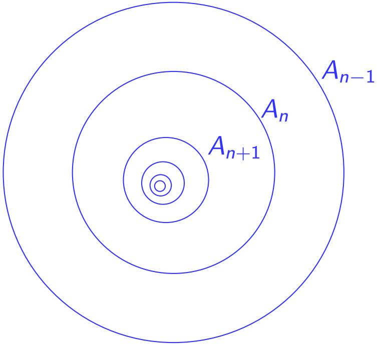
$$0 \leq \left\langle \widehat{\psi}_P, \left(\widehat{H}(P, \nabla E(P)) - E(P) \right) \widehat{\psi}_P \right\rangle$$
$$\leq \liminf_{n \to \infty} \left\langle W(g_{P,n})\psi_{P,n}, \left(\widehat{H}(P, \nabla E(P)) - E(P) \right) W(g_{P,n})\psi_{P,n} \right\rangle = 0,$$

which finishes the proof.

An Abstract Compactness Result for Fock spaces

For $d \in \mathbb{N}$ consider a sequence $(A_n)_{n\in\mathbb{N}}$ of open sets $A_n \subset \mathbb{R}^d$ satisfying

$$A_n \supset A_{n+1}, \quad \text{dist}(A_{n+1}, A_n^c) > 0 \quad \text{for all } n \in \mathbb{N} \quad \text{and} \quad \text{vol}\Big(\bigcap_{n\in\mathbb{N}} A_n\Big) = 0. \tag{2}$$



Theorem 10 (Matte, H-Hinrichs-Siebert)

Let $M \subset \mathcal{F}$ and assume the following.

(a) *There exists $g \in L^2(\mathbb{R}^d)$ such that $\|a_k\psi\| \leq |g(k)|$ for almost all $k \in \mathbb{R}^d$ and all $\psi \in M$.*

(b) *$(A_n)_{n\in\mathbb{N}}$ satisfies (2). For all $n \in \mathbb{N}$, we have, as a function of $q \in \mathbb{R}^d$,*

$$\lim_{q\to 0} \sup_{\psi\in M} \int_{A_n^c} \|a_{k+q}\psi - a_k\psi\|^2 dk = 0.$$

Then M is relatively compact.

The proof is based on the following two results.

Theorem 11 (Frechet 1908)

Let $\mathcal{H} = \bigoplus_{n=1}^{\infty} \mathfrak{h}_n$ with $\mathfrak{h}_n$ being a Hilbert space for all $n \in \mathbb{N}$. Then $M \subset \mathcal{H}$ is relatively compact if and only if the following two conditions are satisfied.

(i) *For each $n \in \mathbb{N}$ there exists a compact subset $K_n \subset \mathfrak{h}_n$ such that for all $f = (f_n)_{n \in \mathbb{N}} \in M$ we have $f_n \in K_n$.*

(ii) $\sup_{f \in M} \sum_{k=n}^{\infty} \|f_k\|^2 \overset{n \to \infty}{\longrightarrow} 0$.

Theorem 12 (Kolmogorov 1931, Riesz 1933, Sudakov 1957)

A subset $M \subset L^2(\mathbb{R}^m)$ is relatively compact if and only if

(i) $\lim_{h \to 0} \sup_{f \in M} \|f(\cdot + h) - f(\cdot)\| = 0$,

(ii) $\lim_{R \to \infty} \sup_{f \in M} \|1_{B_R(0)^c} f\| = 0$.

Applying the abstract compactness Theorem

Define

$$R_n(P,k) := (H_n(P-k) - E_n(P) + \omega_n(k))^{-1}. \quad (3)$$

We will use a so called pull-through relation.

Lemma 13 (Fröhlich,...)

Let $P \in \mathcal{I}_n$. Then $a_k W(g_{P,n})\psi_{P,n} \in \mathcal{F}$ for almost all $k \in \mathbb{R}^d$ and

$$a_k W(g_{P,n})\psi_{P,n} = (-v(k)R_n(P,k) + g_{P,n}(k))\psi_{P,n}.$$

Lemma 14

Assme $P \in \mathcal{I}_n$. Then

$$\|R_n(P,k)\| \leq \frac{\Delta_{P,n}}{\omega_n(k)} \quad \textit{for all } k \in \mathbb{R}^d.$$

Analytic Perturbation Theory

Since $E_n(P)$ is a (simple) discrete eigenvalue for $P \in \mathcal{I}_n$ we can apply analytic perturbation theory in the total momentum to calculate its derivatives.

Lemma 15

Let $P \in \mathcal{I}_n$. Then $\xi \mapsto E_n(\xi)$ is analytic for ξ in a neighborhood of P and the following holds:

- $\partial_i E_n(P) = \langle \psi_{P,n}, \partial_i \Theta(P - P_{\mathrm{f}}) \psi_P \rangle$ *for all* $i = 1, \ldots, d$.
- *For all* $i = 1, \ldots, d$,

$$\partial_i^2 E_n(P) \leq C - 2 \left\| (H_n(P) - E_n(P))^{-1/2} \left(\partial_i \Theta(P - P_{\mathrm{f}}) - \partial_i E_n(P) \right) \psi_{P,n} \right\|^2 .$$

for some constant C independent of n.

Infrared Bounds

Lemma 16

Let $P \in \mathcal{I}_n$ and $|\nabla E_n(P)| < 1$. Then there exists a constant $C > 0$ (independent of n) such that

$$\|R_n(P,k)(\partial_i \Theta(P - P_f) - \partial_i E_n(P)\psi_{P,n}\| \quad (4)$$

$$\leq \sqrt{\frac{(\Delta_{P,n} + \Delta_{P,n}^2)(1+|k|)}{\omega_n(k)}} \left| 1 - \frac{1}{2}\partial_i^2 E_n(P) \right|^{1/2} \quad \textit{for all } k \in \mathbb{R}^d. \quad (5)$$

Proof.

$$\text{L.H.S} \leq \underbrace{\|R_n(P,k)^{1/2}\|}_{\leq \sqrt{\Delta_{P,n}/\omega_n(k)}} \underbrace{\|R_n(P,k)^{1/2}(H_n(P) - E_n(P))^{1/2}\|}_{\substack{\sqrt{(1+|k|)(1+\Delta_{P,n})} \\ \text{bounded since difference is relatively bounded}}}$$

$$\underbrace{\|(H_n(P) - E_n(P))^{-1/2}(\partial_i \Theta(P - P_f) - \partial_i E_n(P))\psi_{P,n}\|}_{\substack{\leq |1 - \frac{1}{2}\partial_i^2 E_n(P)|^{1/2} \\ \text{by analytic perturbation theory,} \\ \text{it is the root of the second order derivative of the energy}}}$$

Similarly as the previous Lemma one shows the following lemma.

Lemma 17

Assme $P \in \mathcal{I}_n$. Then for all $k \in \mathbb{R}^d$

$$\left\| \left(R_n(P,k) - \frac{1}{\omega_n(k) - k\nabla E_n(P)} \right) \psi_{P,n} \right\|$$

$$\leq C \sum_{i=1}^{d} (1 + |1 - \frac{1}{2}\partial_i^2 E_n(P)|^{1/2}) \frac{\max\{1, \Delta_{P,n}\}}{1 - |\nabla E_n(P)|} (1 + \omega_n(k)^{-1/2})$$

for some constant C independent of n.

Compactness of the set of approximate Ground States

Theorem 18

Assume

$$P \in \bigcap_n \mathcal{I}_n, \quad \sup_n \Delta_{P,n} < \infty, \quad |\nabla E_n(P)| < 1, \quad \sup_{i,n} |\partial_i^2 E_n(P)| < \infty.$$

Then, with $g_{P,n} := f_{\nabla E_n(P),n} \in L^2(\mathbb{R}^d)$ *the set*

$$\left\{ W(g_{P,n})\psi_{P,n} : n \in \mathbb{N} \right\}$$

is relatively compact.

To prove the theorem we verify the two assumptions of the abstract compactness Theorem in Fock spaces.

- By the pull-through relation and the infrared bounds there exists an n-independent constant $C > 0$ such that

$$\|a_k W(g_{P,n})\psi_{P,n}\| \leq C(1+\omega_n^{-1/2}(k))|v(k)| \quad \text{a.e. } k \in \mathbb{R}^d \ , \ n \in \mathbb{N}.$$

This verifies the first assumption.

- Let $A_m = \{k \in \mathbb{R}^d : |k| < 1/m\}$, $m \in \mathbb{N}$. Then using the pull-through relation and infrared bounds one can show that there exist constants C_1 and C_2 such that

$$\begin{aligned}
&\left(\int_{A_m^c} \left\| a_{k+q} W(g_{P,n})\psi_{P,n} - a_k W(g_{P,n})\psi_{P,n} \right\|^2 dk \right)^{1/2} \\
&\quad \leq C_1 \|v(\cdot + q) - v(\cdot)\|_2 + |q| C_2 \|v\|_2 \\
&\quad + \|1_{A_m^c}(g_{P,n}(\cdot + q) - g_{P,n}(\cdot))\|_2 \\
&\quad \longrightarrow 0
\end{aligned}$$

as $q \to 0$. This verifies the second assumption.