The master Dyson-Schwinger equation and the Wilsonian renormalization group flows in a generally covariant setting

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Outline

I. On Wilsonian regularized Feynman functional integral formulation:

- Can be substituted by regularized master Dyson-Schwinger equation for correlators.
- For conformally invariant or flat spacetime Lagrangians, showed an existence condition for regularized MDS solutions, provides convergent iterative solver method.

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II. On Wilsonian renormalization group flows of correlators:

- They form a topological vector space which is
Hausdorff, locally convex, complete, nuclear, semi-Montel, Schwartz.
- On flat spacetime for bosonic fields: in bijection with distributional correlators.

[`arXiv:2303.03740` *with Zsigmond Tarcsay*]

Part I:

On Wilsonian regularized Feynman functional integral formulation

Followed guidelines

Do not use (unless emphasized):

- Structures specific to an affine spacetime manifold.
- Known fixed spacetime metric / causal structure.
- Known splitting of Lagrangian to free + interaction term.

Consequences:

- Cannot go to momentum space, have to stay in spacetime description.
- Cannot refer to any affine property of Minkowski spacetime, e.g. asymptotics. (No Schwartz functions.)
- Cannot use Wick rotation to Euclidean signature metric.
- Even if Wick rotated, no free + interaction splitting, so no Gaussian Feynman measure.
- Can only use generic, differential geometrically natural objects.

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Test field variations: $\mathcal{D}$, the compactly supported ones from $\mathcal{E}$ with $\mathcal{D}$ test function topology.

Informal Feynman functional integral in Lorentz signature

Fix a reference field $\psi_0 \in \boldsymbol{\mathcal{E}}$ for bringing the problem from $\boldsymbol{\mathcal{E}}$ to $\mathcal{E}$, and take $J_1, ..., J_n \in \mathcal{E}'$. Then, $\psi \mapsto (J_1|\psi-\psi_0) \cdot ... \cdot (J_n|\psi-\psi_0)$ defines a $\boldsymbol{\mathcal{E}} \to \mathbb{R}$ polynomial observable.

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Feynman type quantum vacuum expectation value of this is postulated as:

$$\int_{\psi\in\boldsymbol{\mathcal{E}}} (J_1|\psi-\psi_0) \cdot \ldots \cdot (J_n|\psi-\psi_0) \;\; \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)} \, \mathrm{d}\lambda(\psi) \;\Bigg/\; \int_{\psi\in\boldsymbol{\mathcal{E}}} \mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)} \, \mathrm{d}\lambda(\psi)$$

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Partition function often invoked to book-keep these (formal Fourier transform of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$):

$$Z_{\psi_0}: \quad \mathcal{E}' \longrightarrow \mathbb{C}, \quad J \longmapsto Z_{\psi_0}(J) := \int\limits_{\psi\in\boldsymbol{\mathcal{E}}} \mathrm{e}^{\mathrm{i}\,(J|\psi-\psi_0)}\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S(\psi)}\,\mathrm{d}\lambda(\psi),$$

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$$G^{(n)}_{\psi_0} \quad := \quad \left((-\mathrm{i})^n \, \frac{1}{Z_{\psi_0}(J)} \, \partial^{(n)}_J Z_{\psi_0}(J) \right)\Big|_{J=0}$$

n-field correlator, and their collection $G_{\psi_0} := \left(G^{(0)}_{\psi_0}, G^{(1)}_{\psi_0}, \ldots, G^{(n)}_{\psi_0}, \ldots\right) \in \bigoplus_{n\in\mathbb{N}_0} \overset{n}{\otimes} \mathcal{E}$.

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Above quantum expectation value expressable via distribution pairing: $\left(J_1\otimes\ldots\otimes J_n \,\middle|\, G^{(n)}_{\psi_0}\right)$.

Well known problems:

- No “Lebesgue” measure λ in infinite dimensions.
- Neither $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$ is meaningful. (Can be repaired to some extent in Euclidean signature.)
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Textbook "theorem": because of above rules, one has
$Z:\,\mathcal{E}'\to\mathbb{C}$ is Fourier transform of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S}\,\lambda$ "$\Longleftrightarrow$" it satisfies master-Dyson-Schwinger eq

$$\Big(\,\mathbf{E}\big((-\mathrm{i})\partial_J+\psi_0\big)\;+\;\hbar\,J\,\Big)Z(J)\quad=\quad 0\quad(\forall J\in\mathcal{E}')$$

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Does this informal PDE have a meaning? [Yes, on the correlators $G=\big(G^{(0)},G^{(1)},\ldots\big)$.]

Rigorous definition of Euler-Lagrange functional

- Let a Lagrange form be given, which is

$$\mathrm{L}: \; V(\mathcal{M}) \;\oplus\; T^*(\mathcal{M}) {\otimes} V(\mathcal{M}) \;\oplus\; T^*(\mathcal{M}) {\wedge} T^*(\mathcal{M}) {\otimes} V(\mathcal{M}) {\otimes} V^*(\mathcal{M}) \;\longrightarrow\; \overset{\dim(\mathcal{M})}{\wedge} T^*(\mathcal{M})$$

pointwise bundle homomorphism.

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where $S_{\mathcal{K}}(v,\nabla):=\int_{\mathcal{K}}\mathrm{L}(v,\nabla v,F(\nabla))$ for all $\mathcal{K}\subset\mathcal{M}$ compact.

Action functional $S : \boldsymbol{\mathcal{E}} \to \mathrm{Meas}(\mathcal{M}, \mathbb{R})$ Fréchet differentiable, its Fréchet derivative

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  More precisely, they sit in a graded-symmetrized subspace, e.g. $\bigvee(\mathcal{E})$ or $\bigwedge(\mathcal{E})$ of $\mathcal{T}(\mathcal{E})$.
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  Unity $\mathbb{1}:=(1,0,0,0,\dots)$.
  Left-multiplication L_x by a fix element x meaningful and continuous linear.
  Left-insertion ι_p (tracing out) by $p\in\big(\mathcal{T}(\mathcal{E})\big)'\equiv\mathcal{T}_a(\mathcal{E}')$ also meaningful, continuous linear.
  Usual graded-commutation: $\big(\,\iota_p\,L_{\delta\psi}\,\pm\,L_{\delta\psi}\,\iota_p\,\big)\,G\;=\;(p|\delta\psi)\,G\quad(\forall p\in\mathcal{E}',\;\delta\psi\in\mathcal{E},\;G\,)$.

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We say that O is multipolynomial iff for some $\psi_0 \in \boldsymbol{\mathcal{E}}$ there exists $\mathbf{O}_{\psi_0} \in \mathcal{T}_a(\mathcal{E}')$, such that

$$\forall\, \psi \in \boldsymbol{\mathcal{E}}: \quad \underbrace{O_{\psi_0}(\psi{-}\psi_0)}_{=\,O(\psi)} \;=\; \Big(\mathbf{O}_{\psi_0}\,\Big|\,\big(1,\, \overset{1}{\otimes}(\psi{-}\psi_0),\, \overset{2}{\otimes}(\psi{-}\psi_0),\, \ldots\big)\Big).$$

Similarly $E : \boldsymbol{\mathcal{E}} \to \mathcal{D}'$, $\psi \mapsto E(\psi)$, let $E_{\psi_0} := E \circ (\mathrm{I}_{\mathcal{E}} + \psi_0)$ the same re-expressed on $\mathcal{E}$.

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$\iota_{(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)}$ meaningfully acts on $\mathcal{T}(\mathcal{E})$.

The master Dyson-Schwinger (MDS) equation is:

we search for (ψ_0, G_{ψ_0}) such that:

$$\underbrace{G^{(0)}_{\psi_0}}_{=:\, b\, G_{\psi_0}} \quad = \quad 1,$$

$$\forall\, \delta\psi_T \in \mathcal{D}: \qquad \underbrace{\Big(\, \boldsymbol{\iota}_{(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)} \;-\; \mathrm{i}\,\hbar\, L_{\delta\psi_T} \,\Big)}_{=:\, \mathbf{M}_{\psi_0,\,\delta\psi_T}}\, G_{\psi_0} \quad = \quad 0.$$

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[Feynman type quantum vacuum expectation value of O is then $\big(\mathbf{O}_{\psi_0} \,\big|\, G_{\psi_0}\big)$.]

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$$E: \quad \boldsymbol{\mathcal{E}} \times \mathcal{D} \longrightarrow \mathbb{R}, \quad (\psi,\, \delta\psi_T) \longmapsto \int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\, \Box_y \psi(y)\, \mathrm{v}(y) \;+\; \int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\, \psi^3(y)\, \mathrm{v}(y).$$

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$$\Downarrow$$

MDS operator at $\psi_0 = 0$ reads

$$\big(\mathbf{M}_{\psi_0,\delta\psi_T}\, G\big)^{(n)}(x_1, \ldots, x_n) =$$

$$\int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\, \Box_y G^{(n+1)}(y, x_1, \ldots, x_n)\, \mathrm{v}(y) \;+\; \int\limits_{y\in\mathcal{M}} \delta\psi_T(y)\, G^{(n+3)}(y, y, y, x_1, \ldots, x_n)\, \mathrm{v}(y)$$

$$-\mathrm{i}\,\hbar \underbrace{n\, \tfrac{1}{n!} \textstyle\sum_{\pi\in\Pi_n} \delta\psi_T(x_{\pi(1)})\, G^{(n-1)}\big(x_{\pi(2)}, \ldots, x_{\pi(n)}\big)}_{=\,(L_{\delta\psi_T} G)^{(n)}(x_1,\ldots,x_n)}$$

Pretty much well-defined, and clear recipe, if field correlators were *functions*.

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Workaround in QFT: Wilsonian regularization using coarse-graining (UV damping).

Wilsonian regularized master Dyson-Schwinger equation

- When $\mathcal{E}$ (resp $\mathcal{D}$) are smooth sections of some vector bundle, denote by $\mathcal{E}^{\times}$ (resp $\mathcal{D}^{\times}$) the smooth sections of its densitized dual vector bundle. Then, distributional sections are $\mathcal{D}^{\times\prime}$ (resp $\mathcal{E}^{\times\prime}$).

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Coarse-graining ops are natural generalization of convolution by test functions to manifolds.

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Brings back problem from distributions to smooth functions, but depends on regulator κ.

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But what we do with κ dependence? (Rigorous Wilsonian renormalization?)

Part II:

On Wilsonian RG flows of correlators

Informal Wilsonian RG flows of Feynman measures

Fix a reference field $\psi_0 \in \boldsymbol{\mathcal{E}}$ to bring the problem from $\boldsymbol{\mathcal{E}}$ to $\mathcal{E}$.

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$\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\psi_0,C_\nu}}\lambda_{C_\nu}$ is the pushforward of $\mathrm{e}^{\frac{\mathrm{i}}{\hbar}S_{\psi_0,C_\kappa}}\lambda_{C_\kappa}$ by C_μ. $\longleftarrow$ RGE

Rigorous definition will be this, but expressed on the formal moments (n-field correlators).

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For any distributional correlator G, the family

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Space of Wilsonian RG flows naturally topologized by coarse-graining-wise Tychonoff.

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Space of Wilsonian RG flows is nonempty:

For any distributional correlator G, the family

$$\mathcal{G}^{(n)}_{C_\kappa} \quad := \quad \otimes^n C_\kappa \, G^{(n)} \qquad (*)$$

is a Wilsonian RG flow.

Space of Wilsonian RG flows naturally topologized by coarse-graining-wise Tychonoff.

Theorem[A.L., Z.Tarcsay]:

1. On manifolds it is Hausdorff, locally convex, complete, nuclear, semi-Montel, Schwartz.
2. On flat spacetime for bosonic fields, all Wilsonian RG flows are of the form of $(*)$.

Sketch of proof for 1.:

- Coarse-grainings have a natural partial ordering of being less UV than an other: $C_\nu \preceq C_\kappa$ iff $C_\nu = C_\kappa$ or $\exists\, C_\mu:\ C_\nu = C_\mu C_\kappa$.
- With this, the space of Wilsonian RG flows is seen to be projective limit of copies of $\mathcal{T}(\mathcal{E})$.
- Check known properties of $\mathcal{T}(\mathcal{E})$, some of them are preserved by projective limit.

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Sketch of proof for 2.:

- On flat spacetime, convolution ops by test functions $C_f := f \star (\cdot)$ exist and commute.
- Due to RGE, commutativity of convolution ops, and polarization formula for n-forms, for bosonic fields $\mathcal{G}^{(n)}_{C_f}$ is n-order homogeneous polynomial in f.

  That is, one has corresponding $\mathcal{G}^{(n)}_{f_1,\dots,f_n}$ symmetric n-linear map in $f_1, ..., f_n$.
- Due to RGE, commutativity of convolution ops, and an improved Banach-Steinhaus thm, $\mathcal{G}^{(n)}_{f_1^t,\dots,f_n^t}\Big|_0$ extends to an n-variate distribution, which will do the job.

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An improved Banach-Steinhaus theorem (the key lemma – A.L., Z.Tarcsay):
If a sequence of n-variate distributions pointwise converge on $\otimes^n\mathcal{D}$, it does also on full $\mathcal{D}_n$.
Therefore, by ordinary Banach-Steinhaus thm, the limit is an n-variate distribution.

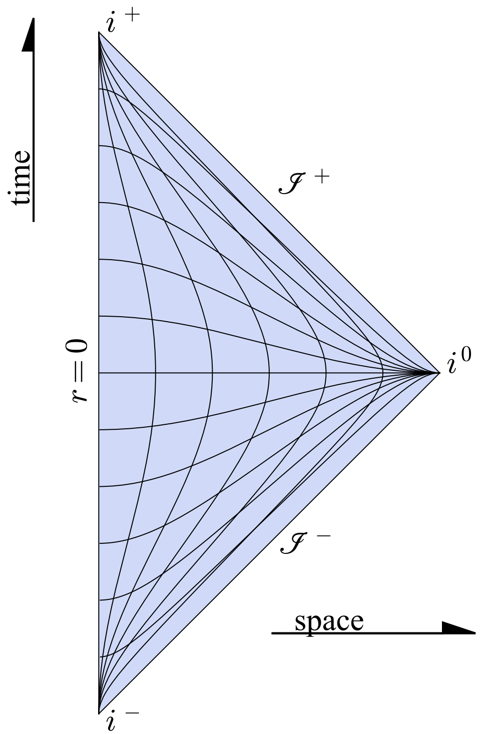
Summary

- Feynman integral has no rigorous definition in Lorentz signature.
- Can be substituted by master Dyson-Schwinger (MDS) equation.
- Function spaces and operators for MDS equation are well defined (in suitable variables).
- Wilsonian regularized version of MDS equation is well defined (in suitable variables).
- Does not need a pre-arranged fixed causal structure.
- Existence condition proved for Wilsonian regularized MDS solutions. Provides a convergent iterative approximation algorithm.
- Space of Wilsonian RG flows of correlators:
  - Hausdorff, locally convex, complete, nuclear, semi-Montel, Schwartz.
  - are in bijection with distributions, on flat spacetime for bosonic fields.

Backup slides

Existence condition for regularized MDS solutions

If Euler-Lagrange functional $E : \boldsymbol{\mathcal{E}} \to \mathcal{D}'$ conformally invariant:
re-expressable on Penrose conformal compactification.



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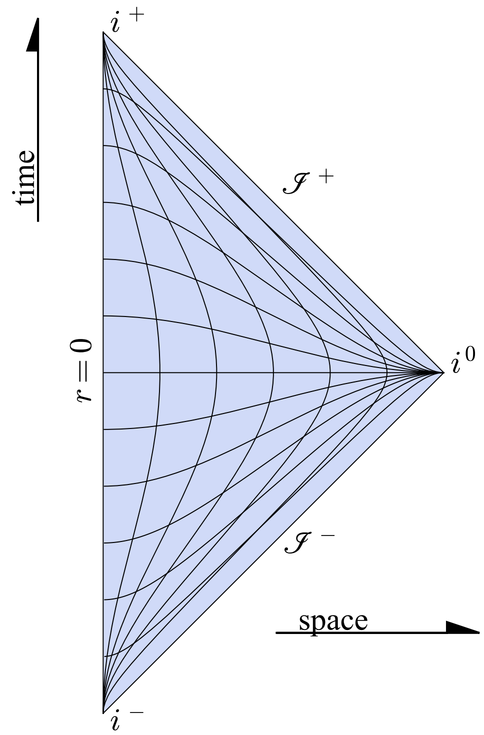

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$E : \boldsymbol{\mathcal{E}} \to \mathcal{D}'$ reformulable over this base manifold.

In such situation, $\mathcal{E} = \mathcal{D}$ and have nice properties:
countably Hilbertian nuclear Fréchet (CHNF) space.

$$F_0 \supset F_1 \supset \ldots \supset F_m \supset \ldots \supset \mathcal{E}$$

(Intersection of shrinking Hilbert spaces F_m with Hilbert-Schmidt embedding.)

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Theorem [Dubin,Hennings:*P.RIMS*__25__(1989)971]:
without penalty, one can equip $\mathcal{T}(\mathcal{E})$ with a better topology, inheriting CHNF topology.

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Regularized MDS operator is then a Hilbert-Schmidt linear map

$$\mathbf{M}_{\psi_0,\kappa} : \quad H_m \otimes F_m \longrightarrow H_0, \quad \mathcal{G} \otimes \delta\psi_T \longmapsto \mathbf{M}_{\psi_0,\kappa,\delta\psi_T}\mathcal{G}$$

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Theorem: one can legitimately trace out $\delta\psi_T$ variable to form

$$\hat{\mathbf{M}}^2_{\psi_0,\kappa} : \quad H_m \longrightarrow H_m, \quad \mathcal{G} \longmapsto \sum_{i \in \mathbb{N}_0} \mathbf{M}^\dagger_{\psi_0,\kappa,\delta\psi_T\,i} \mathbf{M}_{\psi_0,\kappa,\delta\psi_T\,i} \mathcal{G}$$

By construction: $\mathcal{G}$ is κ-regularized MDS solution $\iff$ $b\,\mathcal{G} = 1$ and $\hat{\mathbf{M}}^2_{\psi_0,\kappa}\mathcal{G} = 0$.

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(i) the iteration

$$\mathcal{G}_0 \;:=\; \mathbb{1} \;\text{ and }\; \mathcal{G}_{l+1} \;:=\; \mathcal{G}_l - \tfrac{1}{T}\,\hat{\mathbf{M}}^2_{\psi_0,\kappa}\mathcal{G}_l \qquad (l = 0, 1, 2, ...)$$

is always convergent if $T >$ trace norm of $\hat{\mathbf{M}}^2_{\psi_0,\kappa}$.

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Use for lattice-like numerical method in Lorentz signature?
(Treatment can be adapted to flat spacetime also, because Schwartz functions are CHNF.)

Fréchet derivative in top.vector spaces

Let F and G real top.affine space, Hausdorff.
Subordinate vector spaces: $\mathbb{F}$ and $\mathbb{G}$.

A map $S:\ F \to G$ is Fréchet-Hadamard differentiable at $\psi \in F$ iff:
there exists $DS(\psi):\ \mathbb{F} \to \mathbb{G}$ continuous linear, such that for all sequence $n \mapsto h_n$ in $\mathbb{F}$, and nonzero sequence $n \mapsto t_n$ in $\mathbb{R}$ which converges to zero,

$$(\mathbb{G})\lim_{n\to\infty}\left(\frac{S(\psi+t_n\,h_n)-S(\psi)}{t_n}-DS(\psi)\,h_n\right) \quad = \quad 0$$

holds.

Fréchet derivative of action functional

Fréchet derivative of $S:\,\boldsymbol{\mathcal{E}}\longrightarrow\mathrm{Meas}(\mathcal{M},\mathbb{R})$ is

$$DS:\,\boldsymbol{\mathcal{E}}\times\mathcal{E}\longrightarrow\mathrm{Meas}(\mathcal{M},\mathbb{R}),\;(\psi,\delta\psi)\longmapsto\Big(\mathcal{K}\mapsto\big(DS_{\mathcal{K}}(\psi)\big|\delta\psi\big)\Big)$$

For $\underbrace{(v,\nabla)}_{=:\psi}\in\boldsymbol{\mathcal{E}}$ given,

$$\underbrace{(\delta v,\delta C)}_{=:\delta\psi}\;\mapsto\;\big(DS_{\mathcal{K}}(v,\nabla)\,\big|\,(\delta v,\delta C)\big)=$$

$$\int\limits_{\mathcal{K}}\Big(D_1\mathrm{L}(v,\nabla v,P(\nabla))\,\delta v+D_2^a\mathrm{L}(v,\nabla v,P(\nabla))\,(\nabla_a\delta v+\delta C_a v)+2\,D_3^{[ab]}\mathrm{L}(v,\nabla v,P(\nabla))\,\tilde{\nabla}_{[a}\delta C_{b]}\Big)$$

$$=\int\limits_{\mathcal{K}}\Big(D_1\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\ldots c_m]}\,\delta v-\big(\tilde{\nabla}_a D_2^a\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\ldots c_m]}\big)\,\delta v\Big)+$$

$$\Big(D_2^a\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\ldots c_m]}\,\delta C_a v-2\,\big(\tilde{\nabla}_a D_3^{[ab]}\mathrm{L}(v,\nabla v,P(\nabla))_{[c_1\ldots c_m]}\big)\,\delta C_b\Big)$$

$$+\;m\int\limits_{\partial\mathcal{K}}\Big(D_2^a\mathrm{L}(v,\nabla v,P(\nabla))_{[ac_1\ldots c_{m-1}]}\,\delta v+2\,D_3^{[ab]}\mathrm{L}(v,\nabla v,P(\nabla))_{[ac_1\ldots c_{m-1}]}\,\delta C_b\Big)$$

$$(m:=\dim(\mathcal{M}))$$

[usual Euler-Lagrange bulk integral + boundary integral]

Distributions on manifolds

$W(\mathcal{M})$ vector bundle, $W^{\times}(\mathcal{M}) := W^{*}(\mathcal{M}) \otimes \overset{\dim(\mathcal{M})}{\wedge} T^{*}(\mathcal{M})$ its densitized dual. $W^{\times\times}(\mathcal{M}) \equiv W(\mathcal{M})$.

Correspondingly: $\mathcal{E}^{\times}$ and $\mathcal{D}^{\times}$ are densitized duals of $\mathcal{E}$ and $\mathcal{D}$.

$\mathcal{E} \times \mathcal{D}^{\times} \to \mathbb{R}$, $(\delta\psi, p_{T}) \mapsto \int\limits_{\mathcal{M}} \delta\psi\, p_{T}$ and $\mathcal{D} \times \mathcal{E}^{\times} \to \mathbb{R}$, $(\delta\psi_{T}, p) \mapsto \int\limits_{\mathcal{M}} \delta\psi_{T}\, p$ jointly sequentially continuous.

Therefore, continuous dense linear injections $\mathcal{E} \to \mathcal{E}^{\times\prime}$ and $\mathcal{D} \to \mathcal{D}^{\times\prime}$.
(hance the name, distributional sections)

Let $A:\ \mathcal{E} \to \mathcal{E}$ continuous linear.
It has formal transpose iff there exists $A^{t}:\ \mathcal{D}^{\times} \to \mathcal{D}^{\times}$ continuous linear, such that $\forall \delta\psi \in \mathcal{E}$ and $p_{T} \in \mathcal{D}^{\times}$: $\int\limits_{\mathcal{M}} (A\, \delta\psi)\, p_{T} = \int\limits_{\mathcal{M}} \delta\psi\, (A^{t}\, p_{T})$.
Topological transpose of formal transpose $(A^{t})'$: $(\mathcal{D}^{\times})' \to (\mathcal{D}^{\times})'$ is the distributional extension of A. Not always exists.

Fundamental solution on manifolds

Let $E:\ \boldsymbol{\mathcal{E}}\times\mathcal{D}\to\mathbb{R}$ be Euler-Lagrange functional, and $J\in\mathcal{D}'$.

$\mathtt{K}_{(J)}\in\boldsymbol{\mathcal{E}}$ is solution with source J, iff $\forall\delta\psi_T\in\mathcal{D}:\ \left(E(\mathtt{K}_{(J)})\,|\,\delta\psi_T\right)=(J|\delta\psi_T)$.

Specially: one can restrict to $J\in\mathcal{D}^\times\subset\mathcal{E}^\times\subset\mathcal{D}'$.

A continuous map $\mathtt{K}:\ \mathcal{D}^\times\to\boldsymbol{\mathcal{E}}$ is fundamental solution, iff for all $J\in\mathcal{D}^\times$ the field $\mathtt{K}(J)\in\boldsymbol{\mathcal{E}}$ is solution with source J.

May not exists, and if does, may not be unique.

If $\mathtt{K}_{\psi_0}:\ \mathcal{D}^\times\to\mathcal{E}$ vectorized fundamental solution is linear (e.g. for linear $E_{\psi_0}:\ \mathcal{E}\to\mathcal{D}'$): $\mathtt{K}_{\psi_0}\in\mathcal{L}in(\mathcal{D}^\times,\mathcal{E})\subset(\mathcal{D}^\times)'\otimes(\mathcal{D}^\times)'$ is distribution.

Particular solutions to the free MDS equation

Distributional solutions to free MDS equation: $G_{\psi_0} = \exp(K_{\psi_0})$ where

$$
\begin{aligned}
K^{(0)}_{\psi_0} &= 0, \\
K^{(1)}_{\psi_0} &= 0, \\
K^{(2)}_{\psi_0} &= \mathrm{i}\,\hbar\,\mathtt{K}^{(2)}_{\psi_0} \\
K^{(n)}_{\psi_0} &= 0 \qquad\qquad (n \geq 2)
\end{aligned}
$$

Smooth function solutions to free regularized MDS equation: $G_{\psi_0} = \exp(K_{\psi_0,\kappa})$ where

$$
\begin{aligned}
K^{(0)}_{\psi_0,\kappa} &= 0, \\
K^{(1)}_{\psi_0,\kappa} &= 0, \\
K^{(2)}_{\psi_0,\kappa} &= \mathrm{i}\,\hbar\,(C_\kappa \otimes C_\kappa)\mathtt{K}^{(2)}_{\psi_0} \\
K^{(n)}_{\psi_0,\kappa} &= 0 \qquad\qquad (n \geq 2)
\end{aligned}
$$

[Here $C_\kappa(\cdot) := \eta \star (\cdot)$ is convolution by a test function η.]

Renormalization from functional analysis p.o.v.

Let $\mathbb{F}$ and $\mathbb{G}$ real or complex top.vector space, Hausdorff loc.conv complete.

Let $M : \mathbb{F} \rightarrowtail \mathbb{G}$ densely defined linear map (e.g. MDS operator).

Closed: the graph of the map is closed.

Closable: there exists linear extension, such that its graph closed (unique if exists).

Closable $\Leftrightarrow$ where extendable with limits, it is unique.

Multivalued set:
$\mathrm{Mul}(M) := \big\{ y \in \mathbb{G} \,\big|\, \exists\, (x_n)_{n\in\mathbb{N}}$ in $\mathrm{Dom}(M)$ such that $\lim_{n\to\infty} x_n = 0$ and $\lim_{n\to\infty} M x_n = y \big\}$.

$\mathrm{Mul}(M)$ always closed subspace.

Closable $\Leftrightarrow \mathrm{Mul}(M) = \{0\}$.

Maximally non-closable $\Leftrightarrow \mathrm{Mul}(M) = \overline{\mathrm{Ran}(M)}$. Pathological, not even closable part.

Polynomial interaction term of MDS operator maximally non-closable!

MDS operator:

$$\mathbf{M}: \quad \mathcal{D} \otimes \mathcal{T}(\mathcal{E}) \to \mathcal{T}(\mathcal{E}), \quad G \mapsto \mathbf{M}\, G$$

linear, everywhere defined continuous. So,

$$\mathbf{M}: \quad \mathcal{T}({\mathcal{D}^{\times}}') \rightarrowtail \mathcal{D}' \otimes \mathcal{T}({\mathcal{D}^{\times}}'), \quad G \mapsto \mathbf{M}\, G$$

linear, densely defined.

Similarly: $\mathbf{M}_{\kappa}$ regularized MDS operator (κ: a fix regularizator).

Not good equation:

$$G \in \mathcal{T}(\mathcal{D}^{\times\prime})\,? \qquad G^{(0)} = 1 \text{ and } \exists\, \mathcal{G}_\kappa \to G \text{ approximator sequence, such that :}$$
$$\lim_{\kappa \to \delta} \mathbf{M}\, \mathcal{G}_\kappa = 0.$$

All G would be selected, because $\mathrm{Mul}()$ set of interaction term is full space.

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Can be good:

$$G \in \mathcal{T}(\mathcal{D}^{\times\prime})\,? \qquad G^{(0)} = 1 \text{ and } \exists\, \mathcal{G}_\kappa \to G \text{ approximator sequence, such that :}$$
$$\forall\,\kappa:\ \mathbf{M}_\kappa\,\mathcal{G}_\kappa = 0.$$

That is, as implicit function of κ, not as operator closure kernel.

Running coupling:
If in $\mathbf{M}_\kappa$ EL terms are combined with κ-dependent weights $\gamma(\kappa)$.
(Not just with real factors.)
E.g.:

$$(\gamma, G) \in \mathcal{T}(\mathcal{D}^{\times\prime})\,? \qquad G^{(0)} = 1 \text{ and } \exists\, \mathcal{G}_\kappa \to G \text{ approximator sequence, such that :}$$
$$\forall\,\kappa:\ \mathbf{M}_{\gamma(\kappa),\kappa}\,\mathcal{G}_\kappa = 0.$$

Feynman integral "$\Longleftrightarrow$" MDS equation.

Wilsonian regularized Feynman integral:
integrate not on $\mathcal{E}$, only on the image space $C_\kappa[\mathcal{E}]$ of a smoothing operator $C_\kappa : \mathcal{E} \to \mathcal{E}$.

[Smoothing operator: $\sim$ convolution, can be generalized to manifolds. Does UV damping.]
Automatically knows RGE relations.

Wilsonian regularized Feynman integral "$\Longleftrightarrow$" regularized MDS equation + RGE:

$$\big(\psi_0, \kappa \mapsto \gamma(\kappa), \kappa \mapsto \mathcal{G}_{\psi_0,\kappa}\big) = ? \text{ such that : } \qquad \underbrace{\mathcal{G}^{(0)}_{\psi_0,\kappa}}_{=:\, b\,\mathcal{G}_{\psi_0,\kappa}} = 1,$$

$$\forall \kappa : \forall\, \delta\psi_T \in \mathcal{D} : \qquad \underbrace{\Big(\iota_{\gamma(\kappa)\,(\mathbf{E}_{\psi_0}\,|\,\delta\psi_T)} - \mathrm{i}\,\hbar\, L_{C_\kappa \delta\psi_T} \Big)}_{=:\, \mathbf{M}_{\psi_0,\kappa,\delta\psi_T}} \mathcal{G}_{\psi_0,\kappa} = 0,$$

$$\text{RGE} \longrightarrow \qquad \forall \mu, \kappa : \quad \mathcal{G}^{(n)}_{\psi_0,(C_\mu \kappa)} = \big(\otimes^n C_\mu\big)\, \mathcal{G}^{(n)}_{\psi_0,\kappa}.$$

Running coupling is meaningful. Conjecture: RG flow of $\mathcal{G}_{\psi_0,\kappa} \leftrightarrow$ distributional G_{ψ_0}.
(Conjecture proved for flat spacetime for bosonic fields.)

Some complications on topological vector spaces

Careful with tensor algebra! Schwartz kernel theorems:

$$\hat{\otimes}^n_\pi \mathcal{E} \equiv \mathcal{E}_n \equiv (\hat{\otimes}^n_\pi \mathcal{E}')' \equiv \mathcal{L}in(\mathcal{E}', \hat{\otimes}^{n-1}_\pi \mathcal{E})$$

$$(\hat{\otimes}^n_\pi \mathcal{E})' \equiv \mathcal{E}'_n \equiv \hat{\otimes}^n_\pi \mathcal{E}' \equiv \mathcal{L}in(\mathcal{E}, \hat{\otimes}^{n-1}_\pi \mathcal{E}')$$

$$\hat{\otimes}^n_\pi \mathcal{D} \underset{\text{cont.bij.}}{\leftarrow} \mathcal{D}_n \equiv (\hat{\otimes}^n_\pi \mathcal{D}')'$$

$$(\hat{\otimes}^n_\pi \mathcal{D})' \rightarrow \mathcal{D}'_n \equiv \hat{\otimes}^n_\pi \mathcal{D}' \equiv \mathcal{L}in(\mathcal{D}, \hat{\otimes}^{n-1}_\pi \mathcal{D}')$$

$\mathcal{E} \times \mathcal{E} \to F$ separately continuous maps are jointly continuous.
$\mathcal{E}' \times \mathcal{E}' \to F$ separately continuous bilinear maps are jointly continuous.
For mixed, no guarantee.
For $\mathcal{D}$ or $\mathcal{D}'$ spaces, joint continuity from separate continuity of bilinear forms not automatic.
For mixed, even less guarantee.

But as convergence vector spaces, everything is nice with mixed $\mathcal{E}$, $\mathcal{E}'$, $\mathcal{D}$, $\mathcal{D}'$ multilinears (separate sequential continuity $\Leftrightarrow$ joint sequential continuity).