Relative Entropy for a scalar field in a Noncommutative Minkowski Spacetime

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Motivation

Is noncommutative QFT physically realistic?

Does noncommutative Geometry award us with a theory of quantum gravity?

Intro, What

- **Calculate relative Entropy** for a QFT in **NC-Minkowski space**

Intro, Why

- Physical (Dis-)Proof for NC Spacetime
- Supply a proof as to the **connection to quantum gravity**

Intro NCQFT - Conceptual Part

QFT in a NC Minkwoski-spacetime is represented on $\mathcal{V} \otimes \mathscr{F}_s(\mathscr{H})$, where $\mathcal{V}$ is the representation space of $\hat{x}$

$$[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu},$$

where $\mu, \nu = 0, \ldots, 3$ and θ is a skew-symmetric (w.r.t. the Minkowski metric) matrix

$$\theta_{\mu\nu} = \begin{pmatrix} 0 & \Theta & 0 & 0 \\ \Theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \Theta' \\ 0 & 0 & -\Theta' & 0 \end{pmatrix},$$

with $\Theta, \Theta' \in \mathbb{R}$ and $\Theta \geq 0$.

Intro NCQFT II

GL07 proved that $\phi_\otimes$ can be written on the Fock space $\mathscr{F}_s(\mathscr{H})$ by existence of a unitary map $\mathcal{U}$ from $\mathcal{U} : \mathcal{V} \otimes \mathscr{F}_s(\mathscr{H}) \to \mathscr{F}_s(\mathscr{H})$

$$\begin{aligned}\phi_\theta(f) := &\int d^4x\, f(x)\, \phi_\theta(x) \\ = &\int \frac{d^3\mathrm{p}}{\omega_\mathrm{p}} \,\left(f^-(p)\, e^{-ip\theta P} a(p) + f^+(p)\, e^{ip\theta P} a^*(p)\right),\end{aligned}$$

with $\omega_p = +\sqrt{\vec{p}^2 + m^2}$ and $p = (\omega_p, \vec{p})$ and $f \in \mathscr{S}(\mathbb{R}^4)$ and P is the momentum operator.

Furthermore, the authors proved that ϕ_θ (and $\phi_{-\theta}$) is a wedge local field.

Twisted CCR

The twisted CCR algebra for arbitrary on-shell momenta $p, p' \in \mathscr{H}_m^+$ and matrices θ, θ' is

$$a_\theta(p)a_{\theta'}(p') = e^{ip(\theta+\theta')p'} a_{\theta'}(p')a_\theta(p)$$
$$a^*_\theta(p)a^*_{\theta'}(p') = e^{ip(\theta+\theta')p'} a^*_{\theta'}(p')a^*_\theta(p)$$
$$a_\theta(p)a^*_{\theta'}(p') = e^{-ip(\theta+\theta')p'} a^*_{\theta'}(p')a_\theta(p) + \omega_{\mathsf{p}}\, \delta^3(p-p')e^{-ip(\theta-\theta')P}$$

Equivalent to Deformation with Warped Convolutions

Definition of deformation, BLS10

The warped convolution A_θ of $A \in \mathcal{C}^\infty$ is given on a set of vectors $\Phi \in \mathcal{D} \subset \mathscr{H}$ by

$$A_\theta \Phi := (2\pi)^{-d} \lim_{\epsilon \to 0} \iint dy\, dk\, \chi(\epsilon x, \epsilon y)\, e^{-iyk}\, U(\theta y)\, A\, U(-\theta y) U(k) \Phi.$$

where $U(k) := e^{ik^\mu P_\mu}$. It is connected to the Rieffel product as follows,

$$A_\theta B_\theta = (A \times_\theta B)_\theta$$

where $\times_\theta$ is known as the Rieffel product on $\mathcal{C}^\infty$.

Crash Course Tomita-Takesaki I

Given $(\mathcal{M}, \mathscr{H}, |\Omega\rangle)$ the Tomita-Takesaki operator S applies to elements of the algebra as follows

$$S\, A\, |\Omega\rangle = A^* |\Omega\rangle$$

S has a polar decomposition denoted by

$$S = J\Delta^{1/2},$$

with J antilinear and unitary, $J^2 = -1$, $J\mathcal{M}J = \mathcal{M}'$ and Δ is self-adjoint and non-negative acting as an automorphism $\Delta^{it}\mathcal{M}\Delta^{-it} = \mathcal{M}$.

Relative Entropy

Given $(\mathcal{M}, \mathscr{H}, |\Omega\rangle)$ + cyclic and separating state ω' ($|\Omega'\rangle$) the **relative** Tomita-Takesaki operator $S_{\omega',\omega}$ applies as

$$S_{\omega',\omega}\, A\, |\Omega'\rangle = A^* |\Omega\rangle$$

S (since closeable) has again polar decomposition denoted by

$$S_{\omega',\omega} = J_{\omega',\omega} \Delta^{1/2}_{\omega',\omega},$$

with $J_{\omega',\omega}$ antilinear and unitary $J^2 = -1$, $J_{\omega',\omega} \mathcal{M} J_{\omega',\omega} = \mathcal{M}'$ and $\Delta_{\omega',\omega}$ is self-adjoint and non-negative.

The **Araki-Uhlmannn** relative entropy is given by

$$S_{rel}(\omega', \omega) = -\langle \Omega | \log(\Delta_{\omega',\omega}) \Omega \rangle$$

Relative Entropy

For the calculation of the relative entropy, five objects are needed.

- 2 cyclic and separating states $\omega(\cdot) = \langle \Omega | \cdot \ \Omega \rangle$ and $\omega'(\cdot) = \langle \Omega' | \cdot \ \Omega' \rangle$
- a von Neumann algebra $\mathcal{M}$, a Hilbert space $\mathscr{H}$
- a modular operator Δ

In case the two states are unitarily equivalent, i.e.

$$\omega(U \ \cdot \ U^{-1}) = \omega'(\cdot)$$

the relative entropy or Araki-Uhlmann entropy formula reduces to

$$S_{rel}(\omega', \omega) = i \frac{d}{dt} \langle \Omega | U \Delta^{it} U^* \Delta^{-it} \Omega \rangle |_{t=0}$$

Relative Entropy II-Example

- Bisognano-Wichmann theorem: Restricting the algebra to the Rindler wedge, i.e. $x_1 > 0$ at $x_0 = 0$ the modular operator is

$$\Delta = e^{2\pi \boldsymbol{L_{01}}},$$

where L_{01} represents the boost generator

$$L_{01} = \int d^{d-1}x \, x^1 \, T_{00}(x),$$

and T_{00} is the 00-component of the energy-momentum tensor.

Relative Entropy III

Let ω be the vacuum state, and ω' generated by $U|\Omega\rangle = e^{i\phi(f)}|\Omega\rangle$ then the relative (or Araki-Uhlmann) entropy is

$$
\begin{aligned}
S_0(\omega', \omega) &= i\frac{d}{dt}\langle\Omega|\Delta^{it}_{\omega',\omega}\Omega\rangle|_{t=0} \\
&= i\frac{d}{dt}\langle\Omega|U\Delta^{it}U^*\Delta^{-it}\Omega\rangle|_{t=0} \\
&= i\frac{d}{dt}\langle\Omega|e^{i\phi(f)}e^{2\pi itL_{\mathbf{01}}}e^{-i\phi(f)}e^{-2\pi itL_{\mathbf{01}}}\Omega\rangle|_{t=0},
\end{aligned}
$$

where the QFs are localized in the right (or reference) wedge

$$
\mathcal{W}_R := \{x = (x_0, x_1, \cdots, x_n \in \mathbb{R}^d : x_1 \geq |x_0|)\}
$$

.

NC Generalization

The relative entropy between the deformed field and the vacuum is given by

$$S_\theta(\omega', \omega) = i \frac{d}{dt} \langle \Omega | \Delta^{it}_{\omega'_\theta, \omega} \Omega \rangle |_{t=0}$$
$$= i \frac{d}{dt} \langle \Omega | e^{i\phi_\theta(f)} e^{2\pi i t L_{\mathbf{01}}} e^{-i\phi_\theta(f)} e^{-2\pi i t L_{\mathbf{01}}} \Omega \rangle |_{t=0},$$

since the modular data is the same for the deformed field as for the undeformed field (Theorem 3.5 in BLS10).

NC Relative Entropy Results - Technical Part

Theorem

The deformed relative modular operator $\Delta_{\omega'_\theta,\omega} = e^{i\phi_\theta(f)}$ *converges in the strong limit* $\lim_{\theta\to 0}$*, to the standard relative modular operator* $\Delta_{\omega',\omega}$*, i.e.*

$$\lim_{\theta\to 0} \Delta_{\omega'_\theta,\omega}\Psi \to \Delta_{\omega',\omega}\Psi,$$

for all $\Psi \in \mathscr{F}_s(\mathscr{H})$*. Hence, the relative entropy for a noncommutative field theory reduces in the commutative limit to the standard relative entropy.*

Problem I

No Weyl Relations $\Rightarrow$ Different approach to calculation of relative entropy $\Rightarrow$ represent the boost operator as,

$$L_{01} = i \int d\mu(k)\, a^*(k) \left(\omega_k \frac{\partial}{\partial k^1} \right) a(k).$$

Since only well behaved CCR are between ϕ_θ and $\phi_{-\theta}$

- Replace the particle creation and annihilation operators in the boost operator by their deformed versions.
- Subtract the remaining term that comes from this deformed replacement.

Problem II

This procedure gives us

$$\begin{aligned} L_{01} &= i \int d\mu(k)\ a^*(k) \left(\omega_k \frac{\partial}{\partial k^1} \right) a(k) \\ &= i \int d\mu(k)\, a^*_{-\theta}(k) \left(\omega_k \frac{\partial}{\partial k^1} \right) a_{-\theta}(k) + B \\ &=: L_{01}^{-\theta} + B, \end{aligned}$$

where the operator B is given by

$$\begin{aligned} B &= L_{01} - i \int d\mu(k)\, a^*_{-\theta}(k) \left(\omega_k \frac{\partial}{\partial k^1} \right) a_{-\theta}(k) \\ &= -\theta^{01} \int d\mu(k)\, a^*(k)\, (\omega_k^2 - k_1^2) a(k) + \theta^{01} (P_0^2 - P_1^2) \end{aligned}$$

Solution

The deformed annihilation operator $a_{-\theta}(k)$ applied on $e^{-i\phi_\theta(f)}|\Omega\rangle$ gives

$$a_{-\theta}(k)e^{i\phi_\theta(f)}|\Omega\rangle = f^+(k)\sum_{n=0}^{\infty}\frac{i^n}{n!}\left(\sum_{m=0}^{n-1}\phi_\theta(f)^{n-1-m}\,U(-2\theta k)\,\phi_\theta(f)^m\right)|\Omega\rangle$$

Proof.

$$a_{-\theta}(k)\exp(i\phi_\theta(f))|\Omega\rangle = \sum_{n=0}^{\infty}\frac{i^n}{n!}a_{-\theta}(k)\,\phi_\theta^n(f)|\Omega\rangle$$

$$= \sum_{n=0}^{\infty}\frac{i^n}{n!}[a_{-\theta}(k),\phi_\theta^n(f)]|\Omega\rangle$$

$$= \sum_{n=1}^{\infty}\frac{i^n}{n!}\left([a_{-\theta}(k),\phi_\theta^{n-1}(f)]\phi_\theta(f) + [a_{-\theta}(k),\phi_\theta(f)]\phi_\theta^{n-1}(f)\right)|\Omega\rangle$$

$$= \sum_{n=1}^{\infty}\frac{i^n}{n!}f^+(k)\left(e^{2ik\theta P}\phi_\theta^{n-1}(f) + \cdots + \phi_\theta^{n-1}(f)e^{2ik\theta P}\right)|\Omega\rangle$$

Ansatz

We solved the expressions for the relative entropy by using an expansion in θ.

Lemma

The deformed field ϕ_θ on the dense domain $\mathcal{D}$ of finite particle vectors, has a series expansion in θ given as follows

$$\phi_\theta(g)\Psi = \left(\phi(g) + \sum_{n=1}^{\infty} \frac{1}{n!}(\theta P)_{|\mu_n|}\phi(\nabla^{|\mu_n|} g)\right)\Psi$$

where $\Psi \in \mathcal{D}$, we use the multi-index notation notation $|\mu_n| = \mu_1 + \cdots + \mu_n$ and $\phi(\nabla g) = \int d^4x\,(\frac{\partial}{\partial x} g(x))\phi(x)$.

Solution II

Theorem

The deformed relative entropy $S_\theta(\omega', \omega)$ is positive up to first order in Θ and is explicitly given by

$$
\begin{aligned}
S_\theta(\omega', \omega) &= -2\pi \, \langle \Omega | e^{i\phi_\theta(f)} L_{01} e^{-i\phi_\theta(f)} \Omega \rangle \\
&= S_0(\omega', \omega) + \frac{8\pi}{3} \Theta \left(\int d\mu(k) \, \omega_k \, |f^+(k)|^2 \right)^2
\end{aligned}
$$

where $S_0(\omega', \omega)$ is the undeformed relative entropy.

Interpretation via Beckenstein Bound - Physical Part

In 2008 a proof of the Bekenstein bound was given for QFT by Casini.

Beckenstein Bound? Beckenstein-Hawking (Black Hole) Formula

$$S_{BH} = \alpha M^2,$$

where $\alpha = 4\pi G$ and assume $M >> m$, with entropy S outside of the black hole, the total entropy is

$$S^- = S_{BH} + S.$$

Dropping m into the black hole the entropy is

$$\begin{aligned} S_{BH+m} &= \alpha(M+m)^2 \\ &\approx \alpha M^2 + 2\alpha Mm \\ &= S_{BH} + 2\alpha Mm \end{aligned}$$

Interpretation via Beckenstein Bound

Since entropy always increases we have the (poor man's) inequality

$$S_{BH+m} - S^{-} \geq 0,$$

From which the following bound follows,

$$S \leq 2\alpha\, Mm.$$

Let R be the largest "radius" of the falling system can still be swallowed, and we identify the energy $E = m$ we obtain the Bekenstein bound, i.e.

$$S \leq 2\pi\, R\, E.$$

Analogously to Casini's derivation we have

$$S \leq 2\pi\, R\, E + S_{\theta,rel} - S_{rel}.$$

The deformed version of the Beckenstein bound

$$S \leq 2\pi\, R\, E + \frac{8\pi}{3}\, \Theta\, m^2.$$

By coefficient comparison of the Beckenstein bound but not neglecting the m^2 term

$$S \leq 2\pi\, R\, E + 4\pi\, G\, m^2$$

we identify Θ with the Planck-length squared l_p^2, i.e. $\Theta = \frac{3}{2} G$.

Conclusion and Outlook

- We found a physical connection of θ to the Planck-length, supporting the validity of NCQFT
- Apply the deformation to relative entropy in spherically symmetric spacetimes as done in KPV21

- [*GL*07] H. Grosse and G. Lechner, "Wedge-Local Quantum Fields and Noncommutative Minkowski Space," JHEP **11** (2007)
- [*BLS*10] D. Buchholz, G. Lechner and S. J. Summers, "Warped Convolutions, Rieffel Deformations and the Construction of Quantum Field Theories," Commun. Math. Phys. **304** (2011), 95-123

Thank you for your attention!