Localising Fermionic (S)PDEs

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47th LQP Workshop

22 IX 2023

[1] based on joint work with Ajay Chandra and Martin Hairer

Fermionic PDE

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\partial_t \varphi &= (\Delta - m^2)\varphi - g \langle \bar{\upsilon}, \upsilon \rangle_{\mathbb{R}^2} - \lambda \varphi^3 + \xi \\
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- Multiplication of Distributions $\implies$ Renormalisation

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- How to solve equation without norm?

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- Algebraic Geometry: Points are (finite dimensional) irreducible representations of your algebra

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- Have to extend the CAR algebra!

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  subject to (anti)-linear and $*$-relations.
- Universal Property: $\forall$ $*$-algebra M $\forall \widehat{\pi} \colon \mathfrak{H} \to M$ linear $\exists! \pi \colon \widehat{\mathfrak{A}}(\mathfrak{H}) \to M$ $*$-algebra morphism extension

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- $\mathcal{A}(b)$ is finite dimensional
- Define

$$\mathfrak{A}(\mathfrak{H}) := \widehat{\mathfrak{A}}(\mathfrak{H}) \Big/ \bigcap_{b \in \mathrm{Gr}(\mathfrak{H})} \ker \pi_b$$

with seminorms

$$\|A\|_n := \sup_{\substack{b \in \mathrm{Gr}(\mathfrak{H}) \\ \dim(b) \leqslant n}} \|\pi_b(A)\|$$

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- Under certain conditions one can extend F to certain unbounded elements of $\mathscr{A}(\mathfrak{H})$ to be unbounded operators associated with a von Neumann completion of $\mathcal{A}(\mathfrak{H})$.

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- For each $n \in \mathbb{N}$ obtain maximal local existence time T_n. If $\inf_n T_n = T > 0$, solution exists in $\mathscr{A}(\mathfrak{H})$.

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- Use with more models

Thank You!

References

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