Renewal approach for the energy-momentum relation of the Fröhlich Polaron

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The Polaron

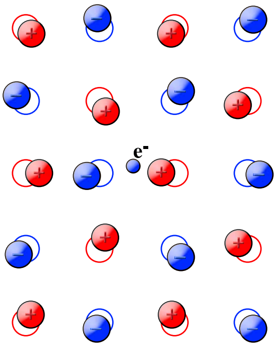

Figure: Source: commons.wikimedia.org (User: Olivier d'ALLIVY KELLY)

Interaction of electron with a polar crystal. The electron drags a cloud of polarization along and thus appears heavier.

The Fröhlich Hamiltonian at fixed total momentum

- Fröhlich Hamiltonian $H(P)$ at fixed total momentum $P \in \mathbb{R}^3$ given by

$$H(P) = \frac{1}{2}(P - P_f)^2 + \mathbf{N} + \frac{\sqrt{\alpha}}{\sqrt{2}\pi}\left(a^*(|\cdot|^{-1}) + a(|\cdot|^{-1})\right)$$

acting on the bosonic Fock space $\mathcal{F}$ over $L^2(\mathbb{R}^3)$, where $\mathbf{N}$ is the number operator, P_f is the momentum operator of the field, $\alpha > 0$ is the coupling constant and

$$(a^*(g)\psi)(k_1, \dots, k_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} g(k_i)\psi(k_1, \dots, \not{k_i}, \dots, k_n)$$

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- Energy-momentum relation:

$$E(P) := \inf \operatorname{spec}(H(P))$$

Energy-momentum relation at the origin

- Energy momentum relation has asymptotics

$$E(P) - E(0) = \frac{1}{2m_{\text{eff}}}|P|^2 + o(|P|^2).$$

as $P \to 0$ where the effective mass m_{eff} is by definition the inverse curvature at $P = 0$.

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- Asymptotics of $E(0)$ in the strong coupling limit given by Pekars conjecture

$$E(0) \sim -c_1\alpha^2, \tag{1}$$

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$$m_{\text{eff}} \sim c_2\alpha^4$$

as $\alpha \to \infty$. Still open!

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- $\liminf_{\alpha\to\infty} m_{\text{eff}}(\alpha)/(\alpha^4 \log(\alpha)^6) > 0$ (Sellke 22)

Energy-momentum relation: global behaviour

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- E has a strict global minimum in 0 (Lampart, Mitrouskas, Mysliwy 22)

Results

Theorem (P. 22)

The following holds.

1. $P \mapsto E(P)$ *is non-decreasing on* $[0, \infty)$ *and strictly increasing on* $\mathcal{I}_0$. *In particular,* $\mathcal{I}_0$ *is an (potentially unbounded) interval.*
2. $P \mapsto E(\sqrt{P})$ *is strictly concave on* $\mathcal{I}_0$. *In particular*
$$E(P) - E(0) < \frac{1}{2m_{eff}} P^2$$
for all $P > 0$, *i.e. the correction to the quasi-particle energy is negative and* $\left[0, \sqrt{2m_{eff}}\right) \subset \mathcal{I}_0$.
3. *For* $|P| \notin \mathrm{cl}(\mathcal{I}_0)$ *we have* $\lim_{\lambda \uparrow E(P)} \langle \Omega, (H(P) - \lambda)^{-1} \Omega \rangle < \infty$, *where* Ω *is the Fock vacuum, in particular* $H(P)$ *does not have a ground state.*

Path measure of the Fröhlich Polaron

- Feynman-Kac formula leads to

$$\langle \Omega, e^{-TH(P)}\Omega \rangle = \int \mathcal{W}(\mathrm{d}X)\, e^{-\mathrm{i}P\cdot X_T} \exp\left(\frac{\alpha}{2} \iint_{[0,T]^2} \mathrm{d}s\mathrm{d}t\, \frac{e^{-|t-s|}}{|X_{s,t}|} \right)$$

where $\mathcal{W}$ is the distribution of a three dimensional Brownian motion and $X_{s,t} := X_t - X_s$.

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  where $\mathcal{W}$ is the distribution of a three dimensional Brownian motion and $X_{s,t} := X_t - X_s$.
- Normalizing by dividing by $\langle \Omega, e^{-TH(0)}\Omega \rangle$ leads to path measure of the Fröhlich Polaron.

Path measure of the Fröhlich Polaron

- Assume that the distribution of $X_T/\sqrt{T}$ under the path measure converges to $\mathcal{N}(0, \sigma^2 I_3)$ as $T \to \infty$. Then $\sigma^2 = m_{\text{eff}}^{-1}$ (Spohn 87, Dybalski, Spohn 20)

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- **Sketch of proof**: For $P \in \mathbb{R}^3$ let Ψ_P denote the ground state of $H(P)$. For $P_T := \frac{1}{\sqrt{T}} P$ and large T

$$\frac{\langle \Omega, e^{-TH(P_T)}\Omega \rangle}{\langle \Omega, e^{-TH(0)}\Omega \rangle} \approx \frac{|\langle \Omega, \Psi_{P_T} \rangle|^2}{|\langle \Omega, \Psi_0 \rangle|^2} e^{-T(E(P_T)-E(0))} \approx e^{-\frac{1}{2m_{\text{eff}}}P^2}$$

and

$$\mathbb{E}_{\mathbb{P}_{\alpha,T}}[e^{-iP_T \cdot X_T}] \approx e^{-\frac{1}{2}\sigma^2 P_0^2}.$$

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- The CLT holds true (Mukherjee, Varadhan 19), (Betz, P. 21)

Point process representation

Modify the point process representation of Mukherjee and Varadhan of the path measure to the matrix element. Set

$$\nu_T(\mathrm{d}s\mathrm{d}t) := \alpha e^{-|t-s|}\mathbb{1}_{\{0\le s<t\le T\}}\mathrm{d}s\mathrm{d}t.$$

and $f_P(T) := \langle \Omega, e^{-TH(P)}\Omega \rangle$. Then

$$\begin{aligned} f_P(T) &= \int \mathcal{W}(\mathrm{d}X)\, e^{-\mathrm{i}P\cdot X_{0,T}} \exp\left(\int \nu_T(\mathrm{d}s\mathrm{d}t)|X_{s,t}|^{-1}\right) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \int \nu_T^{\otimes n}(\mathrm{d}s\mathrm{d}t) \int \mathcal{W}(\mathrm{d}X)\, e^{-\mathrm{i}P\cdot X_{0,T}} \prod_{i=1}^{n} |X_{s_i,t_i}|^{-1} \\ &= e^{c_T} \int \Gamma_T(\mathrm{d}\xi) F_P(T,\xi) \end{aligned}$$

Point process representation

$$f_P(T) = e^{c_T} \int \Gamma_T(\mathrm{d}\xi) F_P(T, \xi)$$

where Γ_T is the distribution of a Poisson point process with intensity measure ν_T, $c_T := \nu_T(\mathbb{R}^2)$ and

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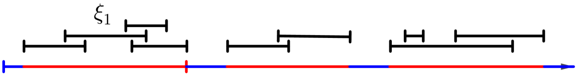

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- F_P decomposes into a product over the distinct clusters

Applying this, yields

$$f_P(T) = e^{\alpha T}\mathbb{E}_\Xi\Big[\mathbb{1}_{\{T_1\leq T\}}F_P(T_1,\xi_1)f_P(T-T_1)\Big] + e^{-P^2T/2}$$

where Ξ is the distribution of the first cluster ξ_1 with length T_1.

Now: Further simplyfy $F_P(T, \xi_1)$. Using the integral identity

$$\frac{1}{|x|} = \sqrt{\frac{2}{\pi}} \int_{[0,\infty)} \mathrm{d}u \, e^{-u^2|x|^2/2}$$

gives

$$F_P(T, \xi) = \int \mathcal{W}(\mathrm{d}X) \, e^{-\mathrm{i}P\cdot X_T} \prod_{i=1}^{n} |X_{s_i,t_i}|^{-1}$$
$$= \int_{[0,\infty)^n} \mathrm{d}u (2/\pi)^{n/2} \int \mathcal{W}(\mathrm{d}X) \, e^{-\mathrm{i}P\cdot X_{0,t}} \, e^{-\sum_{i=1}^{n} u_i^2 |X_{s_i,t_i}|^2/2}$$

There exist a (non-probability) measure μ on marked clusters and a real valued function σ^2 on marked clusters such that

Proposition (P. 22)

The function $T \mapsto f_P(T) := \langle \Omega, e^{-TH(P)} \Omega \rangle$ *satisfies the renewal equation*

$$f_P(T) = \mu\big(e^{-P^2\sigma^2/2} f_P(T - T_1) \mathbb{1}_{\{T_1 \le T\}}\big) + e^{-P^2 T/2}$$

holds for all $P \in \mathbb{R}^3$ *and* $T \geq 0$.

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Now: Take Laplace transform on both sides and use the convolution property of the Laplace transform. Solve for LT of f_P (which is the matrix element of the resolvent).

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Proposition (P. 22)

We have $\mu(e^{-\sigma^2 P^2/2 + E(P)T_1}) \le 1$ *for all* $P \in \mathbb{R}^3$ *and for* $\lambda < E(P)$

$$\langle \Omega, (H(P) - \lambda)^{-1}\Omega \rangle = \frac{1}{P^2/2 - \lambda} \cdot \frac{1}{1 - \mu(e^{-P^2\sigma^2/2 + \lambda T_1})}.$$

If $|P| \in \mathcal{I}_0$ *then* $E(P)$ *is the unique real number satisfying*

$$\mu(e^{-P^2\sigma^2/2 + E(P)T_1}) = 1.$$

Some potential applications

- Closure of the spectral gap in the limit: Show that

$$\forall \varepsilon > 0 \, \exists P \geq 0 : \; \mu(e^{-P^2\sigma^2/2+(E_{\text{ess}}(0)-\varepsilon)T_1}) < \infty$$

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- Potential generalizations to similar models

Thank you for your attention!