BV-BFV formalism: a blueprint for semi-local quantum physics

Kasia Rejzner[1]

University of York

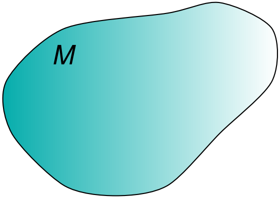
Poznań, 22.09.2023

[1] based on joint works with Klaus Fredenhagen and Michele Schiavina

Outline of the talk

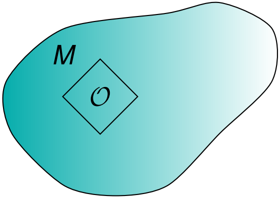
Locality

- The word locality can have different meanings in (A)QFT:



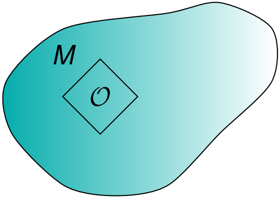
Locality

- The word locality can have different meanings in (A)QFT:
  - Locality as localization, meaning that observables should be localized in bounded regions $\mathcal{O} \subset \mathbb{M}$.



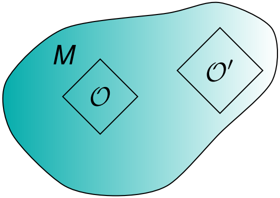
Locality

- The word locality can have different meanings in (A)QFT:
  - Locality as localization, meaning that observables should be localized in bounded regions $\mathcal{O} \subset \mathbb{M}$.
  - Locality in the sense that interactions are point-localized (classically described by local functionals).



Locality

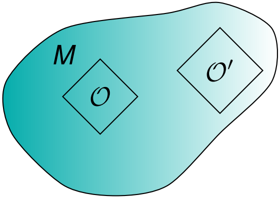
- The word locality can have different meanings in (A)QFT:
  - Locality as localization, meaning that observables should be localized in bounded regions $\mathcal{O} \subset \mathbb{M}$.
  - Locality in the sense that interactions are point-localized (classically described by local functionals).
  - Locality as causality, meaning that observables assigned to spacelike separated regions have to commute (this is the key notion of locality in AQFT).



- The first type of locality fails already for some observables in QED: string-like, wedge-like or cone-like localization.

Locality

- The word locality can have different meanings in (A)QFT:
  - Locality as localization, meaning that observables should be localized in bounded regions $\mathcal{O} \subset \mathbb{M}$.
  - Locality in the sense that interactions are point-localized (classically described by local functionals).
  - Locality as causality, meaning that observables assigned to spacelike separated regions have to commute (this is the key notion of locality in AQFT).



- The first type of locality fails already for some observables in QED: string-like, wedge-like or cone-like localization.
- The second type of locality breaks down if we consider non-local interactions.

Beyond locality

- The third type of locality, however, does not make sense if observables happen to be all localized in all of M (as is the case in gravity for relational observables).

Beyond locality

- The third type of locality, however, does not make sense if observables happen to be all localized in all of M (as is the case in gravity for relational observables).
- Another situation where we need to go beyond locality is when we consider manifolds with boundaries and corners or “boundary at infinity.”

Beyond locality

- The third type of locality, however, does not make sense if observables happen to be all localized in all of M (as is the case in gravity for relational observables).
- Another situation where we need to go beyond locality is when we consider manifolds with boundaries and corners or “boundary at infinity.”
- Many physically interesting instances of non-local theories/observables admit description in terms of appropriate bulk, boundary and corner data.

Beyond locality

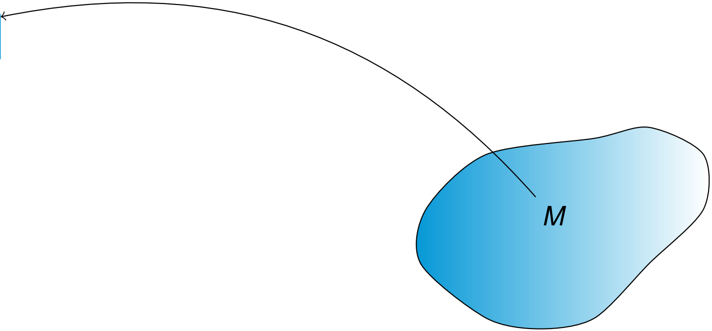
- The third type of locality, however, does not make sense if observables happen to be all localized in all of M (as is the case in gravity for relational observables).
- Another situation where we need to go beyond locality is when we consider manifolds with boundaries and corners or “boundary at infinity.”
- Many physically interesting instances of non-local theories/observables admit description in terms of appropriate bulk, boundary and corner data.
- Question: What is the natural extension of Haag-Kastler axioms (or something similar in spirit) to the situation with boundary and corners (semi-local quantum physics?).

Beyond locality

- The third type of locality, however, does not make sense if observables happen to be all localized in all of M (as is the case in gravity for relational observables).
- Another situation where we need to go beyond locality is when we consider manifolds with boundaries and corners or “boundary at infinity.”
- Many physically interesting instances of non-local theories/observables admit description in terms of appropriate bulk, boundary and corner data.
- Question: What is the natural extension of Haag-Kastler axioms (or something similar in spirit) to the situation with boundary and corners (semi-local quantum physics?).
- Hint: look at the BV-BFV framework, [Cattaneo, Mnev, Reshetikhin, CMP 2011, CMP 2015]

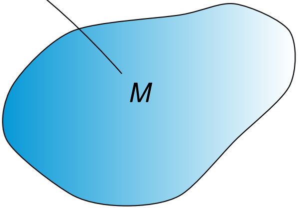
BV-BFV data, following Cattaneo, Mnev, Reshetikhin (CMR)

- $(\mathcal{F}, \Omega, S, Q)$



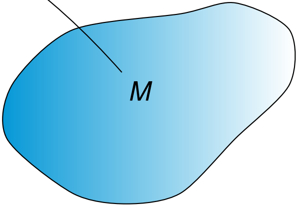
BV-BFV data, following Cattaneo, Mnev, Reshetikhin (CMR)

- $(\mathcal{F}, \Omega, S, Q)$
  - (-1)-symplectic graded manifold $(\mathcal{F}, \Omega)$.



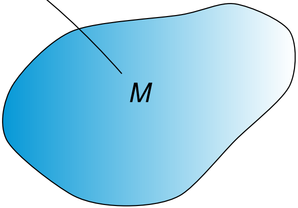
BV-BFV data, following Cattaneo, Mnev, Reshetikhin (CMR)

- $(\mathcal{F}, \Omega, S, Q)$
  - (-1)-symplectic graded manifold $(\mathcal{F}, \Omega)$.
  - Degree 0 action functional S



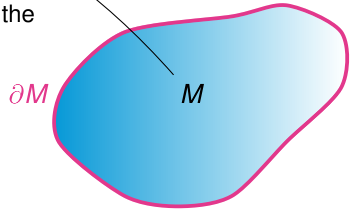
BV-BFV data, following Cattaneo, Mnev, Reshetikhin (CMR)

- $(\mathcal{F}, \Omega, S, Q)$
  - (-1)-symplectic graded manifold $(\mathcal{F}, \Omega)$.
  - Degree 0 action functional S
  - An odd vector field Q on $\mathcal{F}$ of degree 1 with the cohomological property $[Q, Q] = 0$.



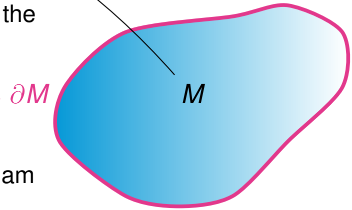
BV-BFV data, following Cattaneo, Mnev, Reshetikhin (CMR)

- $(\mathcal{F}, \Omega, S, Q)$
  - (-1)-symplectic graded manifold $(\mathcal{F}, \Omega)$.
  - Degree 0 action functional S
  - An odd vector field Q on $\mathcal{F}$ of degree 1 with the cohomological property $[Q, Q] = 0$.
- $(\mathcal{F}^\partial, \Omega^\partial, S^\partial, Q^\partial)$



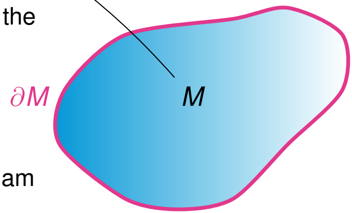
BV-BFV data, following Cattaneo, Mnev, Reshetikhin (CMR)

- $(\mathcal{F}, \Omega, S, Q)$
  - (-1)-symplectic graded manifold $(\mathcal{F}, \Omega)$.
  - Degree 0 action functional S
  - An odd vector field Q on $\mathcal{F}$ of degree 1 with the cohomological property $[Q, Q] = 0$.
- $(\mathcal{F}^\partial, \Omega^\partial, S^\partial, Q^\partial)$
  - Exact (0)-symplectic graded manifold $(\mathcal{F}^\partial, \Omega^\partial = \delta\alpha^\partial)$, where δ denotes the de Rham differential on the space of local forms,



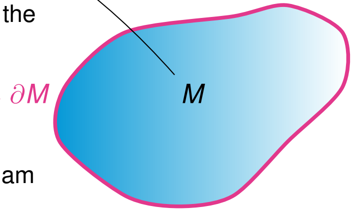
BV-BFV data, following Cattaneo, Mnev, Reshetikhin (CMR)

- $(\mathcal{F}, \Omega, S, Q)$
  - (-1)-symplectic graded manifold $(\mathcal{F}, \Omega)$.
  - Degree 0 action functional S
  - An odd vector field Q on $\mathcal{F}$ of degree 1 with the cohomological property $[Q, Q] = 0$.
- $(\mathcal{F}^\partial, \Omega^\partial, S^\partial, Q^\partial)$
  - Exact (0)-symplectic graded manifold $(\mathcal{F}^\partial, \Omega^\partial = \delta\alpha^\partial)$, where δ denotes the de Rham differential on the space of local forms,
  - Degree 1 local action functional S^∂ on $\mathcal{F}^\partial$,



BV-BFV data, following Cattaneo, Mnev, Reshetikhin (CMR)

- $(\mathcal{F}, \Omega, S, Q)$
  - (-1)-symplectic graded manifold $(\mathcal{F}, \Omega)$.
  - Degree 0 action functional S
  - An odd vector field Q on $\mathcal{F}$ of degree 1 with the cohomological property $[Q, Q] = 0$.
- $(\mathcal{F}^\partial, \Omega^\partial, S^\partial, Q^\partial)$
  - Exact (0)-symplectic graded manifold $(\mathcal{F}^\partial, \Omega^\partial = \delta\alpha^\partial)$, where δ denotes the de Rham differential on the space of local forms,
  - Degree 1 local action functional S^∂ on $\mathcal{F}^\partial$,
  - Odd vector field Q^∂ on $\mathcal{F}^\partial$ of degree 1 with the property: $[Q^\partial, Q^\partial] = 0$.

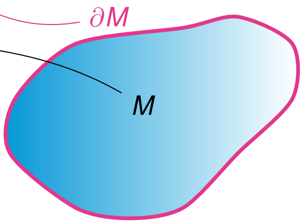


BV-BFV data, following Cattaneo, Mnev, Reshetikhin (CMR)

$(\mathcal{F}, \Omega, S, Q)$ and $(\mathcal{F}^\partial, \Omega^\partial, S^\partial, Q^\partial)$ connected by

$$\pi : \mathcal{F} \to \mathcal{F}^\partial$$

such that:



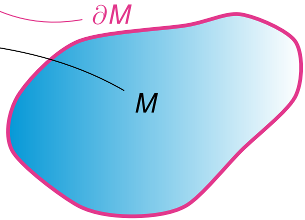
BV-BFV data, following Cattaneo, Mnev, Reshetikhin (CMR)

$(\mathcal{F}, \Omega, S, Q)$ and $(\mathcal{F}^\partial, \Omega^\partial, S^\partial, Q^\partial)$ connected by

$$\pi : \mathcal{F} \to \mathcal{F}^\partial$$

such that:

- $\iota_Q \Omega = \delta S + \pi^* \alpha^\partial$



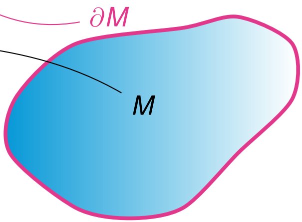
BV-BFV data, following Cattaneo, Mnev, Reshetikhin (CMR)

$(\mathcal{F}, \Omega, S, Q)$ and $(\mathcal{F}^\partial, \Omega^\partial, S^\partial, Q^\partial)$ connected by

$$\pi : \mathcal{F} \to \mathcal{F}^\partial$$

such that:

- $\iota_Q \Omega = \delta S + \pi^* \alpha^\partial$
- $\frac{1}{2}\iota_Q \iota_Q \Omega = \pi^* S^\partial$



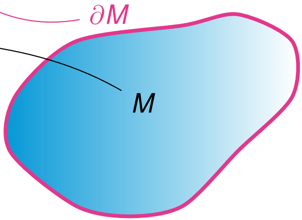
BV-BFV data, following Cattaneo, Mnev, Reshetikhin (CMR)

$(\mathcal{F}, \Omega, S, Q)$ and $(\mathcal{F}^\partial, \Omega^\partial, S^\partial, Q^\partial)$ connected by

$$\pi : \mathcal{F} \to \mathcal{F}^\partial$$

such that:

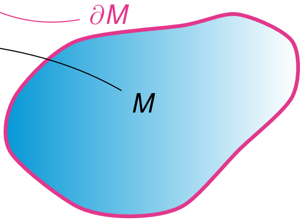
- $\iota_Q \Omega = \delta S + \pi^* \alpha^\partial$
- $\frac{1}{2} \iota_Q \iota_Q \Omega = \pi^* S^\partial$
- $\frac{1}{2} \iota_{Q^\partial} \Omega^\partial = \delta S^\partial$



BV-BFV data, following Cattaneo, Mnev, Reshetikhin (CMR)

$(\mathcal{F}, \Omega, S, Q)$ and $(\mathcal{F}^\partial, \Omega^\partial, S^\partial, Q^\partial)$ connected by

$$\pi : \mathcal{F} \to \mathcal{F}^\partial$$



such that:

- $\iota_Q \Omega = \delta S + \pi^* \alpha^\partial$
- $\frac{1}{2} \iota_Q \iota_Q \Omega = \pi^* S^\partial$
- $\frac{1}{2} \iota_{Q^\partial} \Omega^\partial = \delta S^\partial$
- $\frac{1}{2} \iota_{Q^\partial} \iota_{Q^\partial} \Omega^\partial = 0$

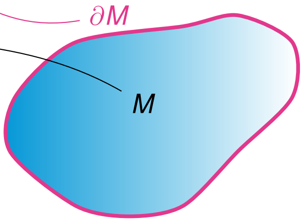
BV-BFV data, following Cattaneo, Mnev, Reshetikhin (CMR)

$(\mathcal{F}, \Omega, S, Q)$ and $(\mathcal{F}^\partial, \Omega^\partial, S^\partial, Q^\partial)$ connected by

$$\pi : \mathcal{F} \to \mathcal{F}^\partial$$

such that:

- $\iota_Q \Omega = \delta S + \pi^* \alpha^\partial$
- $\frac{1}{2}\iota_Q \iota_Q \Omega = \pi^* S^\partial$
- $\frac{1}{2}\iota_{Q^\partial} \Omega^\partial = \delta S^\partial$
- $\frac{1}{2}\iota_{Q^\partial} \iota_{Q^\partial} \Omega^\partial = 0$



We can generalize this and assign data to corners, etc.

Generalization to asymptotic boundary

- Take the limit where ∂M is the boundary "at infinity"

Generalization to asymptotic boundary

- Take the limit where ∂M is the boundary “at infinity”
- Aim: rigorous mathematical description of asymptotic observables and symmetries.

Generalization to asymptotic boundary

- Take the limit where ∂M is the boundary "at infinity"
- Aim: rigorous mathematical description of asymptotic observables and symmetries.
- Asymptotic quantization of QED and QG goes back to Ashtekar (mostly the 80's) and was later developed by others, notably Herdegen (90's to present), who has also been advocating the need to weaken the usual AQFT paradigm of locality.

Generalization to asymptotic boundary

- Take the limit where ∂M is the boundary "at infinity"
- Aim: rigorous mathematical description of asymptotic observables and symmetries.
- Asymptotic quantization of QED and QG goes back to Ashtekar (mostly the 80's) and was later developed by others, notably Herdegen (90's to present), who has also been advocating the need to weaken the usual AQFT paradigm of locality.
- Recent attention: works of Strominger et.al., including *New symmetries of QED* (2015), relate asymptotic charges to the *Weinberg soft photon theorem* and *memory effects*.

Generalization to asymptotic boundary

- Take the limit where ∂M is the boundary "at infinity"
- Aim: rigorous mathematical description of asymptotic observables and symmetries.
- Asymptotic quantization of QED and QG goes back to Ashtekar (mostly the 80's) and was later developed by others, notably Herdegen (90's to present), who has also been advocating the need to weaken the usual AQFT paradigm of locality.
- Recent attention: works of Strominger et.al., including *New symmetries of QED* (2015), relate asymptotic charges to the *Weinberg soft photon theorem* and *memory effects*.
- Asymptotic symmetries in the BV-BFV formalism, Kasia Rejzner, Michele Schiavina, **CMP 2021**.

Physical input

- A globally hyperbolic spacetime M.

Physical input

- A globally hyperbolic spacetime M.
- Configuration space $\mathcal{E}(M)$: choice of objects we want to study in our theory (scalars, vectors, tensors,...).

Physical input

- A globally hyperbolic spacetime M.
- Configuration space $\mathcal{E}(M)$: choice of objects we want to study in our theory (scalars, vectors, tensors,. . .).
- Typically $\mathcal{E}(M)$ is a space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$ over M.

Physical input

- A globally hyperbolic spacetime M.
- Configuration space $\mathcal{E}(M)$: choice of objects we want to study in our theory (scalars, vectors, tensors,...).
- Typically $\mathcal{E}(M)$ is a space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$ over M.
- Classical observables are functionals $F \in \mathcal{C}^\infty(\mathcal{E}(M), \mathbb{R})$, whose derivatives satisfy appropriate regularity conditions.

Physical input

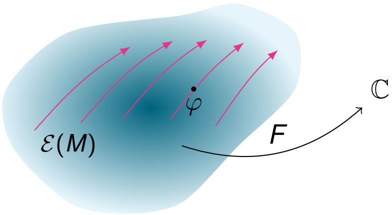
- A globally hyperbolic spacetime M.
- Configuration space $\mathcal{E}(M)$: choice of objects we want to study in our theory (scalars, vectors, tensors,...).
- Typically $\mathcal{E}(M)$ is a space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$ over M.
- Classical observables are functionals $F \in \mathcal{C}^\infty(\mathcal{E}(M), \mathbb{R})$, whose derivatives satisfy appropriate regularity conditions.
- Dynamics: we use a covariant modification of the Lagrangian formalism. Since M is non-compact, the action S is not of the form $S = \int \mathcal{L}(\varphi)$ for some Lagrangian density, but a function $\mathcal{C}_c^\infty(M) \ni f \mapsto \int f\mathcal{L}(\varphi)$ that assigns a functional to each cutoff f.

Physical input

- A globally hyperbolic spacetime M.
- Configuration space $\mathcal{E}(M)$: choice of objects we want to study in our theory (scalars, vectors, tensors,...).
- Typically $\mathcal{E}(M)$ is a space of smooth sections of some vector bundle $E \xrightarrow{\pi} M$ over M.
- Classical observables are functionals $F \in \mathcal{C}^\infty(\mathcal{E}(M), \mathbb{R})$, whose derivatives satisfy appropriate regularity conditions.
- Dynamics: we use a covariant modification of the Lagrangian formalism. Since M is non-compact, the action S is not of the form $S = \int \mathcal{L}(\varphi)$ for some Lagrangian density, but a function $\mathcal{C}_c^\infty(M) \ni f \mapsto \int f\mathcal{L}(\varphi)$ that assigns a functional to each cutoff f.
- From S we obtain a 1-form dS on configuration space that gives the equations of motion: $dS(\varphi) = 0$.

Symmetries

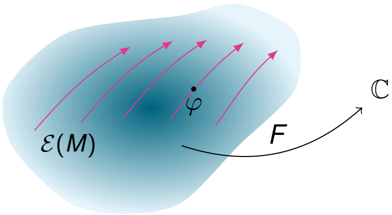
- In the BV framework, symmetries are identified with vector fields (directions) on $\mathcal{E}$.



Symmetries

- In the BV framework, symmetries are identified with vector fields (directions) on $\mathcal{E}$.
- We denote vector fields that are multilocal and compactly supported by $\mathcal{V}$. They act on $\mathcal{F}$ as derivations:

$$\partial_X F(\varphi) := \langle F^{(1)}(\varphi), X(\varphi)\rangle$$

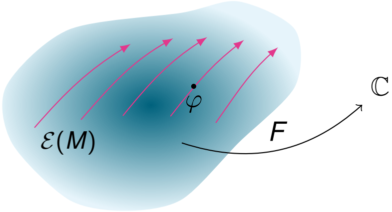


Symmetries

- In the BV framework, symmetries are identified with vector fields (directions) on $\mathcal{E}$.
- We denote vector fields that are multilocal and compactly supported by $\mathcal{V}$. They act on $\mathcal{F}$ as derivations:

$$\partial_X F(\varphi) := \langle F^{(1)}(\varphi), X(\varphi) \rangle$$

- For $X \in \mathcal{V}$ and action S, denote $\langle dS(\varphi), X(\varphi) \rangle \equiv \delta_S(X)(\varphi)$.

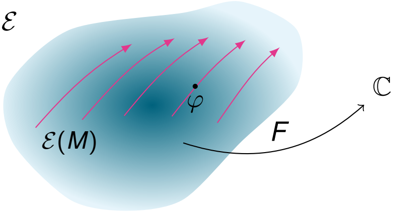


Symmetries

- In the BV framework, symmetries are identified with vector fields (directions) on $\mathcal{E}$.
- We denote vector fields that are multilocal and compactly supported by $\mathcal{V}$. They act on $\mathcal{F}$ as derivations:

$$\partial_X F(\varphi) := \langle F^{(1)}(\varphi), X(\varphi)\rangle$$

- For $X \in \mathcal{V}$ and action S, denote $\langle dS(\varphi), X(\varphi)\rangle \equiv \delta_S(X)(\varphi)$.
- A symmetry of S is a direction in $\mathcal{E}$ in which the action is constant, i.e. it is a vector field $X \in \mathcal{V}$ such that: $\forall \varphi \in \mathcal{E}$: $\delta_S(X) \equiv 0$.



BV complex

- Let the symmetries be characterize by a Lie algebra $\mathfrak{s}$. Extend $\mathcal{E}$ to a graded manifold (extended configuration space) $\overline{\mathcal{E}} \doteq \mathcal{E} \oplus \mathfrak{s}[1]$. The space of functions on $\overline{\mathcal{E}}$ can be equipped with the Chevalley-Elienberg differential γ whose cohomology characterizes the space of gauge-invariant functionals.

BV complex

- Let the symmetries be characterize by a Lie algebra $\mathfrak{s}$. Extend $\mathcal{E}$ to a graded manifold (extended configuration space) $\overline{\mathcal{E}} \doteq \mathcal{E} \oplus \mathfrak{s}[1]$. The space of functions on $\overline{\mathcal{E}}$ can be equipped with the Chevalley-Elienberg differential γ whose cohomology characterizes the space of gauge-invariant functionals.
- The underlying algebra of the BV complex is the space of multivector fields on $\overline{\mathcal{E}}$, i.e. the space of functionals (with appropriate regularity) on the shifted cotangent bundle $\mathcal{F} \equiv T^*[-1]\overline{\mathcal{E}}$ (space of fields). Hence $\mathcal{BV} \subset \mathcal{C}^\infty(T^*[-1]\overline{\mathcal{E}})$.

BV complex

- Let the symmetries be characterize by a Lie algebra $\mathfrak{s}$. Extend $\mathcal{E}$ to a graded manifold (extended configuration space) $\overline{\mathcal{E}} \doteq \mathcal{E} \oplus \mathfrak{s}[1]$. The space of functions on $\overline{\mathcal{E}}$ can be equipped with the Chevalley-Elienberg differential γ whose cohomology characterizes the space of gauge-invariant functionals.
- The underlying algebra of the BV complex is the space of multivector fields on $\overline{\mathcal{E}}$, i.e. the space of functionals (with appropriate regularity) on the shifted cotangent bundle $\mathcal{F} \equiv T^*[-1]\overline{\mathcal{E}}$ (space of fields). Hence $\mathcal{BV} \subset \mathcal{C}^\infty(T^*[-1]\overline{\mathcal{E}})$.
- $\mathcal{BV}$ is equipped with the BV differential $s = \delta_S + \gamma$, which encodes the space of solutions to the equations of motion (in lowest order $\delta_S = -\iota_{dS}$) and the space of invariants under the symmetries.

Antibracket and the Classical Master Equation

- $\mathcal{BV}$, as the space of multivector fields, comes with a shifted Poisson bracket: the Schouten bracket $\{.,.\}$, aka the antibracket.

Antibracket and the Classical Master Equation

- $\mathcal{BV}$, as the space of multivector fields, comes with a shifted Poisson bracket: the Schouten bracket $\{.,.\}$, aka the antibracket.
- Generators of the fibers of $\mathcal{BV}$ are called antifields.

Antibracket and the Classical Master Equation

- $\mathcal{BV}$, as the space of multivector fields, comes with a shifted Poisson bracket: the Schouten bracket $\{.,.\}$, aka the antibracket.
- Generators of the fibers of $\mathcal{BV}$ are called antifields.
- Differential s is not inner with respect to $\{.,.\}$, but locally it can be written as:

$$sX = \{X, S^{\text{ext}}(f)\}\,, \quad f \equiv 1 \text{ on } \operatorname{supp} X$$

Antibracket and the Classical Master Equation

- $\mathcal{BV}$, as the space of multivector fields, comes with a shifted Poisson bracket: the Schouten bracket $\{.,.\}$, aka the antibracket.
- Generators of the fibers of $\mathcal{BV}$ are called antifields.
- Differential s is not inner with respect to $\{.,.\}$, but locally it can be written as:

$$sX = \{X, S^{\text{ext}}(f)\}\,, \quad f \equiv 1 \text{ on } \operatorname{supp} X$$

- ....and S^{ext} is the extended action, which contains ghosts (odd generators of $\overline{\mathcal{E}}$), antifields and potentially more.

Antibracket and the Classical Master Equation

- $\mathcal{BV}$, as the space of multivector fields, comes with a shifted Poisson bracket: the Schouten bracket $\{.,.\}$, aka the antibracket.
- Generators of the fibers of $\mathcal{BV}$ are called antifields.
- Differential s is not inner with respect to $\{.,.\}$, but locally it can be written as:

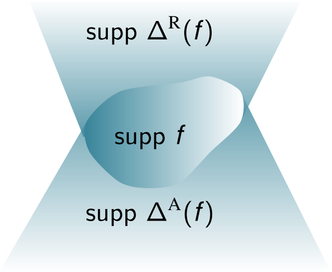
$$sX = \{X, S^{\text{ext}}(f)\}\,, \quad f \equiv 1 \text{ on } \operatorname{supp} X$$

- ....and S^{ext} is the extended action, which contains ghosts (odd generators of $\overline{\mathcal{E}}$), antifields and potentially more.
- The BV differential s has to be nilpotent, i.e.: $s^2 = 0$, which leads to the classical master equation (CME):

$$\{S^{\text{ext}}(f), S^{\text{ext}}(f)\} = 0\,,$$

modulo terms that vanish in the limit of constant f.

Poisson structure

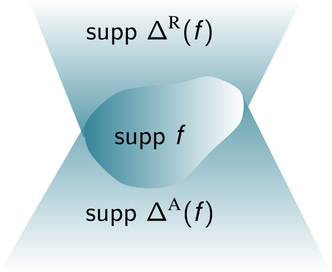


- The (unshifted) Poisson bracket of the free theory is

$$\lfloor F, G \rfloor \doteq \left\langle F^{(1)}, \Delta G^{(1)} \right\rangle ,$$

where $\Delta = \Delta^{\mathrm{R}} - \Delta^{\mathrm{A}}$ is the Pauli-Jordan (commutator) function.

Poisson structure

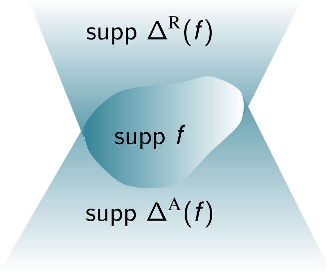


- The (unshifted) Poisson bracket of the free theory is

$$\lfloor F, G \rfloor \doteq \left\langle F^{(1)}, \Delta G^{(1)} \right\rangle ,$$

  where $\Delta = \Delta^{\mathrm{R}} - \Delta^{\mathrm{A}}$ is the Pauli-Jordan (commutator) function.
- For the free scalar field the equation of motion is $P\varphi = 0$, where $P = -(\Box + m^2)$ is (minus) the Klein-Gordon operator.

Poisson structure



- The (unshifted) Poisson bracket of the free theory is

$$\lfloor F, G \rfloor \doteq \left\langle F^{(1)}, \Delta G^{(1)} \right\rangle ,$$

  where $\Delta = \Delta^{\mathrm{R}} - \Delta^{\mathrm{A}}$ is the Pauli-Jordan (commutator) function.
- For the free scalar field the equation of motion is $P\varphi = 0$, where $P = -(\Box + m^2)$ is (minus) the Klein-Gordon operator.
- If M is globally hyperbolic (has a Cauchy surface), P admits retarded and advanced Green's functions Δ^{R}, Δ^{A}.

Deformation of the free theory

- Recall that we can split the action $S = S_0 + V$, where S_0 is quadratic. The free theory (that of S_0) is quantized using deformation quantization.

Deformation of the free theory

- Recall that we can split the action $S = S_0 + V$, where S_0 is quadratic. The free theory (that of S_0) is quantized using deformation quantization.
- Define the $\star$-product (deformation of the pointwise product):

$$(F \star G)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\varphi), W^{\otimes n} G^{(n)}(\varphi) \right\rangle ,$$

where W is the 2-point function of a Hadamard state (on Minkowski spacetime this is just the Wightman 2-point function) and it differs from $\frac{i}{2}\Delta$ by a symmetric bidistribution:

$$W = \frac{i}{2}\Delta + H .$$

Deformation of the free theory

- Recall that we can split the action $S = S_0 + V$, where S_0 is quadratic. The free theory (that of S_0) is quantized using deformation quantization.
- Define the $\star$-product (deformation of the pointwise product):

$$(F \star G)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\varphi), W^{\otimes n} G^{(n)}(\varphi) \right\rangle ,$$

where W is the 2-point function of a Hadamard state (on Minkowski spacetime this is just the Wightman 2-point function) and it differs from $\frac{i}{2}\Delta$ by a symmetric bidistribution:

$$W = \frac{i}{2}\Delta + H .$$

- The free QFT is defined as an appropriate completion of $\mathcal{F}(M)[[\hbar]]$, equipped with $\star$ and the conjugation $*$, where $F^*(\varphi) \doteq \overline{F(\varphi)}$.

Time-ordered product

- Let $\mathcal{F}_{\mathrm{reg}}(M)$ be the space of functionals whose derivatives are test functions, i.e. $F^{(n)}(\varphi) \in \mathcal{D}(M^n)$,

Time-ordered product

- Let $\mathcal{F}_{\mathrm{reg}}(M)$ be the space of functionals whose derivatives are test functions, i.e. $F^{(n)}(\varphi) \in \mathcal{D}(M^n)$,
- The time-ordering operator $\mathcal{T}$ is defined as:

$$\mathcal{T}F(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle F^{(2n)}(\varphi), (\tfrac{\hbar}{2}\Delta^{\mathrm{F}})^{\otimes n} \right\rangle ,$$

where $\Delta^{\mathrm{F}} = \frac{i}{2}(\Delta^{\mathrm{A}} + \Delta^{\mathrm{R}}) + H$ and $H = W - \frac{i}{2}\Delta$.

Time-ordered product

- Let $\mathcal{F}_{\mathrm{reg}}(M)$ be the space of functionals whose derivatives are test functions, i.e. $F^{(n)}(\varphi) \in \mathcal{D}(M^n)$,
- The time-ordering operator $\mathcal{T}$ is defined as:

$$\mathcal{T}F(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle F^{(2n)}(\varphi), (\tfrac{\hbar}{2}\Delta^{\mathrm{F}})^{\otimes n} \right\rangle \ ,$$

where $\Delta^{\mathrm{F}} = \frac{i}{2}(\Delta^{\mathrm{A}} + \Delta^{\mathrm{R}}) + H$ and $H = W - \frac{i}{2}\Delta$.
- Formally it corresponds to the operator of convolution with the oscillating Gaussian measure "with covariance $i\hbar\Delta^{\mathrm{F}}$",

$$\mathcal{T}F(\varphi) \stackrel{\mathrm{formal}}{=} \int F(\varphi - \phi)\, d\mu_{i\hbar\Delta^{\mathrm{F}}}(\phi) \ .$$

Time-ordered product

- Let $\mathcal{F}_{\mathrm{reg}}(M)$ be the space of functionals whose derivatives are test functions, i.e. $F^{(n)}(\varphi) \in \mathcal{D}(M^n)$,
- The time-ordering operator $\mathcal{T}$ is defined as:

$$\mathcal{T}F(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle F^{(2n)}(\varphi), (\tfrac{\hbar}{2}\Delta^{\mathrm{F}})^{\otimes n} \right\rangle ,$$

where $\Delta^{\mathrm{F}} = \frac{i}{2}(\Delta^{\mathrm{A}} + \Delta^{\mathrm{R}}) + H$ and $H = W - \frac{i}{2}\Delta$.

- Formally it corresponds to the operator of convolution with the oscillating Gaussian measure "with covariance $i\hbar\Delta^{\mathrm{F}}$",

$$\mathcal{T}F(\varphi) \stackrel{\mathrm{formal}}{=} \int F(\varphi - \phi)\, d\mu_{i\hbar\Delta^{\mathrm{F}}}(\phi) .$$

- Define the time-ordered product $\cdot_{\mathcal{T}}$ on $\mathcal{F}_{\mathrm{reg}}(M)[[\hbar]]$ by:

$$F \cdot_{\mathcal{T}} G \doteq \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$$

Interaction

- $\cdot_{\mathcal{T}}$ is the time-ordered version of $\star$, in the sense that

$$F \cdot_{\mathcal{T}} G = F \star G\,,$$

if the support of F is later than the support of G.

Interaction

- $\cdot_{\mathcal{T}}$ is the time-ordered version of $\star$, in the sense that

$$F \cdot_{\mathcal{T}} G = F \star G\,,$$

  if the support of F is later than the support of G.
- Interaction is a functional V, for the moment $V \in \mathcal{F}_{\mathrm{reg}}(M)$.

Interaction

- $\cdot_{\mathcal{T}}$ is the time-ordered version of $\star$, in the sense that

$$F \cdot_{\mathcal{T}} G = F \star G\,,$$

  if the support of F is later than the support of G.
- Interaction is a functional V, for the moment $V \in \mathcal{F}_{\mathrm{reg}}(M)$.
- We define the formal S-matrix, $\mathcal{S}(\lambda V) \in \mathcal{F}_{\mathrm{reg}}((\hbar))[[\lambda]]$ by

$$\mathcal{S}(\lambda V) \doteq e_{\mathcal{T}}^{i\lambda V/\hbar} = \mathcal{T}(e^{\mathcal{T}^{-1}(i\lambda V/\hbar)})\,.$$

Interaction

- $\cdot_{\mathcal{T}}$ is the time-ordered version of $\star$, in the sense that

$$F \cdot_{\mathcal{T}} G = F \star G\,,$$

  if the support of F is later than the support of G.
- Interaction is a functional V, for the moment $V \in \mathcal{F}_{\mathrm{reg}}(M)$.
- We define the formal S-matrix, $\mathcal{S}(\lambda V) \in \mathcal{F}_{\mathrm{reg}}((\hbar))[[\lambda]]$ by

$$\mathcal{S}(\lambda V) \doteq e_{\mathcal{T}}^{i\lambda V/\hbar} = \mathcal{T}(e^{\mathcal{T}^{-1}(i\lambda V/\hbar)})\,.$$

- Interacting fields are elements of $\mathcal{F}_{\mathrm{reg}}[[\hbar,\lambda]]$ given by

$$R_{\lambda V}(F) \doteq (e_{\mathcal{T}}^{i\lambda V/\hbar})^{\star-1} \star (e_{\mathcal{T}}^{i\lambda V/\hbar} \cdot_{\mathcal{T}} F) = -i\hbar \frac{d}{d\mu} \mathcal{S}(\lambda V)^{-1} \mathcal{S}(\lambda V + \mu F)\Big|_{\mu=0}$$

Interaction

- $\cdot_{\mathcal{T}}$ is the time-ordered version of $\star$, in the sense that

$$F \cdot_{\mathcal{T}} G = F \star G\,,$$

if the support of F is later than the support of G.

- Interaction is a functional V, for the moment $V \in \mathcal{F}_{\mathrm{reg}}(M)$.
- We define the formal S-matrix, $\mathcal{S}(\lambda V) \in \mathcal{F}_{\mathrm{reg}}((\hbar))[[\lambda]]$ by

$$\mathcal{S}(\lambda V) \doteq e_{\mathcal{T}}^{i\lambda V/\hbar} = \mathcal{T}(e^{\mathcal{T}^{-1}(i\lambda V/\hbar)})\,.$$

- Interacting fields are elements of $\mathcal{F}_{\mathrm{reg}}[[\hbar,\lambda]]$ given by

$$R_{\lambda V}(F) \doteq (e_{\mathcal{T}}^{i\lambda V/\hbar})^{\star-1} \star (e_{\mathcal{T}}^{i\lambda V/\hbar} \cdot_{\mathcal{T}} F) = -i\hbar \frac{d}{d\mu} \mathcal{S}(\lambda V)^{-1} \mathcal{S}(\lambda V + \mu F)\Big|_{\mu=0}$$

- We define the interacting star product as:

$$F \star_V G \doteq R_V^{-1}\left(R_V(F) \star R_V(G)\right)\,,$$

Interaction

- $\cdot_{\mathcal{T}}$ is the time-ordered version of $\star$, in the sense that

$$F \cdot_{\mathcal{T}} G = F \star G ,$$

if the support of F is later than the support of G.
- Interaction is a functional V, for the moment $V \in \mathcal{F}_{\mathrm{reg}}(M)$.
- We define the formal S-matrix, $\mathcal{S}(\lambda V) \in \mathcal{F}_{\mathrm{reg}}((\hbar))[[\lambda]]$ by

$$\mathcal{S}(\lambda V) \doteq e_{\mathcal{T}}^{i\lambda V/\hbar} = \mathcal{T}(e^{\mathcal{T}^{-1}(i\lambda V/\hbar)}) .$$

- Interacting fields are elements of $\mathcal{F}_{\mathrm{reg}}[[\hbar, \lambda]]$ given by

$$R_{\lambda V}(F) \doteq (e_{\mathcal{T}}^{i\lambda V/\hbar})^{\star -1} \star (e_{\mathcal{T}}^{i\lambda V/\hbar} \cdot_{\mathcal{T}} F) = -i\hbar \frac{d}{d\mu} \mathcal{S}(\lambda V)^{-1} \mathcal{S}(\lambda V + \mu F)\Big|_{\mu=0}$$

- We define the interacting star product as:

$$F \star_V G \doteq R_V^{-1} \left(R_V(F) \star R_V(G)\right) ,$$

- Renormalization problem: extend $\cdot_{\mathcal{T}}$, and all the above structures, to V local and non-linear.

QME on regular functionals

- The linearized classical BV operator is defined by

$$s_0 X = \{X, S_0\} .$$

QME on regular functionals

- The linearized classical BV operator is defined by

$$s_0 X = \{X, S_0\}\,.$$

- The quantum master equation is the condition that the S-matrix is invariant under s_0, i.e.: $s_0(\mathcal{S}(V)) = \{e_{\mathcal{T}}^{iV/\hbar}, S_0\} = 0$.

QME on regular functionals

- The linearized classical BV operator is defined by

$$s_0 X = \{X, S_0\} \, .$$

- The quantum master equation is the condition that the S-matrix is invariant under s_0, i.e.: $s_0(\mathcal{S}(V)) = \{e_{\mathcal{T}}^{iV/\hbar}, S_0\} = 0.$
- This expression can be rewritten as:

$$\{e_{\mathcal{T}}^{iV/\hbar}, S_0\} = e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} \left(\frac{1}{2}\{S_0 + V, S_0 + V\} - i\hbar \bigtriangleup (S_0 + V) \right) ,$$

where $\bigtriangleup$ is the BV Laplacian, which on regular functionals is

$$\bigtriangleup X = (-1)^{(1+|X|)} \int dx \frac{\delta^2 X}{\delta\varphi^{\ddagger}(x)\delta\varphi(x)} \, .$$

QME on regular functionals

- The linearized classical BV operator is defined by

$$s_0 X = \{X, S_0\}\,.$$

- The quantum master equation is the condition that the S-matrix is invariant under s_0, i.e.: $s_0(\mathcal{S}(V)) = \{e_{\mathcal{T}}^{iV/\hbar}, S_0\} = 0$.
- This expression can be rewritten as:

$$\{e_{\mathcal{T}}^{iV/\hbar}, S_0\} = e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} \left(\frac{1}{2}\{S_0 + V, S_0 + V\} - i\hbar \triangle (S_0 + V)\right)\,,$$

where $\triangle$ is the BV Laplacian, which on regular functionals is

$$\triangle X = (-1)^{(1+|X|)} \int dx \frac{\delta^2 X}{\delta\varphi^{\ddagger}(x)\delta\varphi(x)}\,.$$

- We obtain the standard form of the QME (as a condition on V):

$$\frac{1}{2}\{S_0 + V, S_0 + V\} = i\hbar \triangle (S_0 + V).$$

Modified QME on a Chauchy slice

- Typically, QME holds on the nose only if we arrange the choices of test functions f in various terms of the action in a specific way.

Modified QME on a Chauchy slice

- Typically, QME holds on the nose only if we arrange the choices of test functions f in various terms of the action in a specific way.
- In general, CME and QME are violated by terms that depend on df.

Modified QME on a Chauchy slice

- Typically, QME holds on the nose only if we arrange the choices of test functions f in various terms of the action in a specific way.
- In general, CME and QME are violated by terms that depend on df.
- Note that replacing f with characteristic function θ of some Cauchy slice (with a compact Cauchy surface), $d\theta$ is supported on the boundary. Hence the support of df plays the role of "smoothed-out boundary."

Modified QME on a Chauchy slice

- Typically, QME holds on the nose only if we arrange the choices of test functions f in various terms of the action in a specific way.
- In general, CME and QME are violated by terms that depend on df.
- Note that replacing f with characteristic function θ of some Cauchy slice (with a compact Cauchy surface), $d\theta$ is supported on the boundary. Hence the support of df plays the role of “smoothed-out boundary.”
- QME has to be replaced by the modified QME

$$s_0\,\mathcal{S}(V) = \mathcal{S}(V) \star \frac{i}{\hbar} R_V(S^{\partial})\,.$$

Modified QME on a Chauchy slice

- Typically, QME holds on the nose only if we arrange the choices of test functions *f* in various terms of the action in a specific way.
- In general, CME and QME are violated by terms that depend on *df*.
- Note that replacing *f* with characteristic function θ of some Cauchy slice (with a compact Cauchy surface), $d\theta$ is supported on the boundary. Hence the support of *df* plays the role of "smoothed-out boundary."
- QME has to be replaced by the modified QME

$$s_0\,\mathcal{S}(V) = \mathcal{S}(V) \star \frac{i}{\hbar} R_V(S^\partial)\,.$$

- Here S^∂ is identified as the BRST charge (compare with [Hollands, **RMP 2007**]) and it is used to select physical states in the Krein-space representation of the BV algebra (similar to CMR). Details will appear in my upcoming paper with Schiavina.

Quantum BV operator I

- The free quantum BV operator $\hat{s}_0$ is defined on regular functionals by:

$$\hat{s}_0 = \mathcal{T}^{-1} \circ s_0 \circ \mathcal{T} = s_0 - i\hbar\triangle\,,$$

Quantum BV operator I

- The free quantum BV operator $\hat{s}_0$ is defined on regular functionals by:

$$\hat{s}_0 = \mathcal{T}^{-1} \circ s_0 \circ \mathcal{T} = s_0 - i\hbar\triangle \,,$$

- The interacting quantum BV operator $\hat{s}$ is defined by:

$$\hat{s} = R_V^{-1} \circ s_0 \circ R_V \,,$$

i.e. it is the twist of the free BV operator by the (non-local!) map that intertwines the free and the interacting theory.

Quantum BV operator I

- The free quantum BV operator $\hat{s}_0$ is defined on regular functionals by:

$$\hat{s}_0 = \mathcal{T}^{-1} \circ s_0 \circ \mathcal{T} = s_0 - i\hbar\triangle \,,$$

- The interacting quantum BV operator $\hat{s}$ is defined by:

$$\hat{s} = R_V^{-1} \circ s_0 \circ R_V \,,$$

  i.e. it is the twist of the free BV operator by the (non-local!) map that intertwines the free and the interacting theory.

- The 0th cohomology of $\hat{s}$ characterizes quantum gauge invariant observables.

Quantum BV operator II

- Assuming QME, we obtain the following expression:

$$\hat{s} = s - i\hbar \triangle \ .$$

Quantum BV operator II

- Assuming QME, we obtain the following expression:

$$\hat{s} = s - i\hbar \triangle \ .$$

- In the presence of boundary terms, we need to correct $\hat{s}$, so that the true BV operator is:

$$\tilde{s} := \hat{s} - \frac{i}{\hbar}[\bullet, S^{\partial}]_{\star_V} = s - i\hbar\triangle\,,$$

which is again local. This again agrees with ideas of CMR.

Thank you very much for your attention!