Hadamard States for Maxwell Fields via Complete Gauge Fixing

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Joint work with Simone Murro (Università di Genova)

Linear Gauge Theories I: Classical Theory

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Definition (Hack-Schenkel 2012; Gérard-Wrochna 2014)

A *linear gauge theory* is a quadruple $(\mathsf{V}_0, \mathsf{V}_1, \mathsf{P}, \mathsf{K})$ consisting of:

(1) two Hermitian bundles $(\mathsf{V}_0, \langle\cdot,\cdot\rangle_{\mathsf{V}_0})$ and $(\mathsf{V}_1, \langle\cdot,\cdot\rangle_{\mathsf{V}_1})$ over M;

(2) a formally self-adjoint linear differential operator $\mathsf{P}\colon \Gamma(\mathsf{V}_1) \to \Gamma(\mathsf{V}_1)$;

(3) a linear differential operator $\mathsf{K}\colon \Gamma(\mathsf{V}_0) \to \Gamma(\mathsf{V}_1)$ s.t.
   - (i) $\mathsf{P} \circ \mathsf{K} = 0$,
   - (ii) $\mathsf{D}_1 := \mathsf{P} + \mathsf{K}\mathsf{K}^* : \Gamma(\mathsf{V}_1) \to \Gamma(\mathsf{V}_1)$ is Green hyperbolic,
   - (iii) $\mathsf{D}_0 := \mathsf{K}^*\mathsf{K}\colon \Gamma(\mathsf{V}_0) \to \Gamma(\mathsf{V}_0)$ is Green hyperbolic.

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Ordinary Field Theories $(\mathsf{K} = 0)$	**Gauge Theories** $(\mathsf{K} \neq 0)$
P hyperbolic fibre metric usually positive-definite	P *non*-hyperbolic fibre metric usually *not* positive-definite

Linear Gauge Theories II: Quantum Theory

Let $(\mathsf{V}_0, \mathsf{V}_1, \mathsf{P}, \mathsf{K})$ be a linear gauge theory on (M, g) and G_1 be the causal propagator of D_1.

$$\mathcal{V}_{\mathrm{P}} := \frac{\ker(\mathsf{K}^*|_{\Gamma_{\mathrm{c}}})}{\mathrm{ran}(\mathsf{P}|_{\Gamma_{\mathrm{c}}})} \xrightarrow[\cong]{[\mathsf{G}_1]} \frac{\ker(\mathsf{P}|_{\Gamma_{\mathrm{sc}}})}{\mathrm{ran}(\mathsf{K}|_{\Gamma_{\mathrm{sc}}})}$$

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Algebraic Quantization:

- **Step 1:** Classical phase space $(\mathcal{V}_{\mathrm{P}}, \sigma([\cdot],[\cdot]) := \mathrm{i}(\cdot, \mathsf{G}_1 \cdot)_{\mathsf{V}_1})$ with $(\cdot,\cdot)_{\mathsf{V}_i} := \int_{\mathsf{M}} \langle \cdot, \cdot \rangle_{\mathsf{V}_i} \operatorname{vol}_g$.

$$(\mathcal{V}_{\mathrm{P}}, \sigma) \qquad \to \qquad \mathrm{CCR}(\mathcal{V}_{\mathrm{P}}, \sigma)$$

$\mathrm{CCR}(\mathcal{V}_{\mathrm{P}}, \sigma) \ldots$ unital $*$-algebra constructed as follows:

generators: $\mathbb{1}, \quad \Phi(v), \quad \Phi^*(v) \quad \forall v \in \mathcal{V}_{\mathrm{P}}$

CCR relations:
$$[\Phi(v), \Phi(w)] = [\Phi^*(v), \Phi^*(w)] = 0$$
$$[\Phi(v), \Phi^*(w)] = \sigma(v,w)\mathbb{1}$$

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- **Step 2:** Construct (quasi-free) **Hadamard State** $\omega\colon \mathrm{CCR}(\mathcal{V}_{\mathrm{P}}, \sigma) \to \mathbb{C}$:

  covariances: $\Lambda^+(v,w) := \omega(\Phi(v)\Phi^*(w)), \qquad \Lambda^-(v,w) := \omega(\Phi^*(w)\Phi(v))$

  Hadamard condition: $\mathrm{WF}'(\lambda^\pm) \subset \mathcal{N}^\pm \times \mathcal{N}^\pm$ where $\Lambda^\pm([s],[t]) =: (s, \lambda^\pm t)_{\mathsf{V}_1}$

  ($\mathcal{N} = \mathcal{N}^+ \cup \mathcal{N}^- \ldots$ light cone in $\mathsf{T}^*\mathsf{M}$)

Known Results

Under some additional assumption on (M, g), Hadamard states for linear gauge theories have been constructed with various different approaches:

Maxwell Theory:

- Furlani (1995) — (M, g) static and Σ compact
- Fewster-Pfenning (2003) — Σ compact and simply-connected
- Dappiaggi-Siemssen (2011) — asymptotically flat spacetimes
- Finster-Strohmaier (2013) — Σ with "absence of zero resonances" condition for Δ_1

Linearized Yang-Mills Theory:

- Hollands (2008) — Σ compact and simply-connected
- Gérard-Wrochna (2014) — Σ compact or $\mathbb{R}^3$

Linearized Gravity:

- Fewster-Hunt (2012), Hunt (2012) — Fock vacuum in Minkowski is Hadamard
- Brunetti-Fredenhagen-Rejzner (2013) — (M, g) ultrastatic and Σ compact
- Benini-Dappiaggi-Murro (2014) — asymptotically flat spacetimes; "radiative" observables
- Gérard-Murro-Wrochna (2022) — partial results

Construction of Hadamard States

$\rho_i\colon \ker(\mathsf{D}_i|_{\Gamma_{\mathrm{sc}}}) \to \Gamma_{\mathrm{c}}(\mathsf{V}_{\rho_i})\ldots$ Cauchy data maps of D_i for suitable bundles V_{ρ_i} over Σ.

$$(\mathcal{V}_{\mathrm{P}},\sigma) \xrightarrow[\cong]{[\rho_1\mathsf{G}_1]} \left(\mathcal{V}_\Sigma := \frac{\ker(\mathsf{K}_\Sigma^\dagger|_{\Gamma_{\mathrm{c}}})}{\mathrm{ran}(\mathsf{K}_\Sigma|_{\Gamma_{\mathrm{c}}})}, \sigma_\Sigma([\cdot],[\cdot]) := \mathrm{i}(\cdot,\mathsf{G}_{1,\Sigma}\cdot)_{\mathsf{V}_{\rho_1}}\right)$$

where:
- $\mathsf{K}_\Sigma := \rho_1\mathsf{K}\mathcal{U}_0$ and $\mathsf{K}_\Sigma^\dagger$ adjoint w.r.t. σ_Σ with $\mathcal{U}_i := \rho_i^{-1}$.
- $\mathsf{G}_{i,\Sigma}\colon \Gamma(\mathsf{V}_{\rho_i}) \to \Gamma(\mathsf{V}_{\rho_i})$ uniquely determined by $\mathsf{G}_i = (\rho_i\mathsf{G}_i)^*\mathsf{G}_{i,\Sigma}(\rho_i\mathsf{G}_i)$.

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Proposition (Gérard-Wrochna 2014; Gérard-Murro-Wrochna 2022)

Suppose $c^\pm : \Gamma_{\mathrm{c}}(\mathsf{V}_{\rho_1}) \to \Gamma(\mathsf{V}_{\rho_1})$ (linear, continuous) are s.t.

(i) $(c^\pm)^\dagger = c^\pm$ and $c^\pm(\mathrm{ran}(\mathsf{K}_\Sigma|_{\Gamma_{\mathrm{c}}})) \subset \mathrm{ran}(\mathsf{K}_\Sigma)$; (Gauge-Invariance)

(ii) $c^+ + c^- = \mathrm{id}$ modulo operator mapping to $\mathrm{ran}(\mathsf{K}_\Sigma)$; (CCR)

(iii) $\pm\sigma_\Sigma(\mathfrak{f}, c^\pm \mathfrak{f}) \geq 0$ for any $\mathfrak{f} \in \ker(\mathsf{K}_\Sigma^\dagger|_{\Gamma_{\mathrm{c}}})$; (Positivity)

(iv) $\mathrm{WF}'(\mathcal{U}_1 c_1^\pm) \subset (\mathcal{N}^\pm \cup F) \times \mathsf{T}^*\Sigma$ where $F \subset \mathsf{T}^*\mathsf{M}\backslash\mathcal{N}$ is conic. (Hadamard)

Then $\lambda^\pm := (\rho_1 \mathsf{G}_1)^*(\pm \mathrm{i}\mathsf{G}_{1,\Sigma}) c^\pm (\rho_1 \mathsf{G}_1)$ defines a quasifree Hadamard state.

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Proposal:

- Fix the gauge degrees of freedom completely:

$$\mathcal{V}_\Sigma := \frac{\ker(\mathsf{K}_\Sigma^\dagger|_{\Gamma_c})}{\operatorname{ran}(\mathsf{K}_\Sigma|_{\Gamma_c})} \xrightarrow[\cong]{\mathsf{T}_\Sigma} \mathcal{V}_\mathsf{R} := \ker(\mathsf{K}_\Sigma^\dagger|_{\Gamma_c}) \cap \ker(\mathsf{R}_\Sigma^\dagger|_{\Gamma_c})$$

  where $\mathsf{R}_\Sigma^\dagger \mathfrak{f} = 0$ is an additional gauge-fixing, s.t.

  (i) no residual gauge freedom.

  (ii) fibre metric on $\mathcal{V}_\mathsf{R}$ is positive.

- Construct state on $\mathcal{V}_\mathsf{R}$ using techniques of Gérard-Wrochna.
- Pulling back with projector T_Σ.

Maxwell's Theory and Cauchy Radiation Gauge

Hermitian bundles $(\mathsf{V}_k, \langle\cdot,\cdot\rangle_{\mathsf{V}_k}):$ $\begin{cases} \mathsf{V}_k := \mathbb{C} \otimes \bigwedge^k \mathsf{T}^*\mathsf{M}, & \Gamma(\mathsf{V}_k) = \Omega^k(\mathsf{M};\mathbb{C}) \\ (\cdot,\cdot)_{\mathsf{V}_k} := \frac{1}{k!}\int_{\mathsf{M}} (g^{-1})^{\otimes k}(\bar{\cdot},\cdot)\,\mathrm{vol}_g \end{cases}$

Maxwell's Theory:

- $(\mathsf{V}_0, \mathsf{V}_1, \mathsf{P}, \mathsf{K})$ with $\mathsf{P} := \delta\mathrm{d}$ and $\mathsf{K} := \mathrm{d}$.
- $\mathsf{D}_i := \Box_i$ where $\Box_i = \delta\mathrm{d} + \mathrm{d}\delta$.
- Canonical gauge condition: $\mathsf{K}^* A = \delta A = 0$. (*Lorenz Gauge*)

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$$(\mathrm{CR}) \quad \Leftrightarrow \quad \underbrace{\delta A = A_0 = 0}_{\textit{radiation gauge}} \quad \overset{\Sigma \text{ non-compact}}{\Leftrightarrow} \quad \underbrace{\delta_\Sigma A_\Sigma = 0}_{\textit{Coulomb Gauge}}$$

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$\hookrightarrow$ $A \in \Omega^1_{\mathrm{sc}}(\mathsf{M}) \Rightarrow$ Find $f \in C^\infty_{\mathrm{sc}}(\mathsf{M})$ s.t. $A' := A + \mathrm{d}f$ satisfies (CR) $\Leftrightarrow$

$$\begin{cases} \Box_0 f = -\delta A \\ \pi = -A_0|_\Sigma \\ \overrightarrow{\Delta}_0 a = -\delta_\Sigma A_\Sigma|_\Sigma \end{cases}$$

with $\overrightarrow{\Delta}_0 = \delta_\Sigma \mathrm{d}_\Sigma$ and $a := f|_\Sigma$, $\pi := \nabla_0 f|_\Sigma$. $\Rightarrow$ Left to solve Poisson equation.

The Poisson Equation on Complete Riemannian Manifolds

$(\Sigma, h) \ldots$ complete and connected Riemannian manifold.

Poisson Equation: $\overrightarrow{\Delta}_0 f = \delta_\Sigma \omega$ for $\omega \in \Omega^1(\Sigma)$ $(*)$

Observation: $(*)$ equivalent to $\omega = \mathrm{d}_\Sigma f + \beta$ for $\beta \in \ker(\delta_\Sigma)$ $\Rightarrow$ **Hodge-type decomposition!**

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  $\hookrightarrow$ For $\omega \in L^2(\mathsf{T}^*\Sigma) \cap \Omega^1(\Sigma)$, $(*)$ has a unique solution (up to constant) on

$$\mathcal{D} := \{f \in C^\infty(\Sigma) \mid \mathrm{d}_\Sigma f \in \overline{\mathrm{d}_\Sigma C^\infty_\mathrm{c}(\Sigma)} \subset L^2(\mathsf{T}^*\Sigma)\}$$

(Proof requires Hodge-Kodaiara decomposition, elliptic regularity and Poincaré duality.)

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  $\hookrightarrow$ In the setting of compactly-supported forms, $(*)$ can only be solved for subspace

$$\Omega^1_\mathsf{H}(\Sigma) := \mathrm{ran}(\mathrm{d}_\Sigma|_{C^\infty_\mathrm{c}}) \oplus \ker(\delta_\Sigma)\,.$$

<u>**Note:**</u> (i) For $\omega \in \Omega^1_\mathsf{H}(\Sigma)$ $(*)$ has unique (up to constant) solution on $C^\infty_\mathrm{c}(\Sigma)$.

(ii) $\Omega^1_\mathsf{H}(\Sigma) = \Omega^1(\Sigma)$ for Σ compact.

Phase Spaces and Complete Gauge Fixing

We call *space of radiation k-forms* $\Omega^k_{\mathsf{R}}(\mathsf{M})$ the subspace of $\Gamma_{\mathrm{sc}}(\mathsf{V}_1) = \Omega^k_{\mathrm{sc}}(\mathsf{M};\mathbb{C})$ defined by

$$\Gamma_{\mathsf{R}}(\mathsf{V}_k) := \begin{cases} \{\omega \in \Omega^k_{\mathrm{sc}}(\mathsf{M};\mathbb{C}) \,|\, \omega_\Sigma|_\Sigma \in \Omega^k_{\mathsf{H}}(\Sigma) \otimes_{\mathbb{R}} \mathbb{C}\} & \text{for } k > 0 \\ C^\infty_{\mathrm{sc}}(\mathsf{M};\mathbb{C}) & \text{for } k = 0 \end{cases},$$

Proposition

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Cauchy data map for Maxwell's theory: $\mathsf{V}_{\rho_1} := \mathsf{V}_1|_\Sigma \oplus \mathsf{V}_1|_\Sigma$

$$\rho_1 \colon \Gamma_{\mathrm{sc}}(\mathsf{V}_1) \to \Gamma_{\mathrm{c}}(\mathsf{V}_{\rho_1}) \cong (C^\infty(\Sigma;\mathbb{C}))^2 \oplus (\Omega^1(\Sigma,\mathbb{C}))^2$$
$$A \mapsto (A_0, \nabla_0 A_0, A_\Sigma, \nabla_0 A_\Sigma)|_\Sigma$$

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Cauchy data map for Maxwell's theory: $\mathsf{V}_{\rho_1} := \mathsf{V}_1|_\Sigma \oplus \mathsf{V}_1|_\Sigma$

$$\rho_1 \colon \Gamma_{\mathrm{sc}}(\mathsf{V}_1) \to \Gamma_{\mathrm{c}}(\mathsf{V}_{\rho_1}) \cong (C^\infty(\Sigma;\mathbb{C}))^2 \oplus (\Omega^1(\Sigma,\mathbb{C}))^2$$
$$A \mapsto (A_0, \nabla_0 A_0, A_\Sigma, \nabla_0 A_\Sigma)|_\Sigma$$

$$\mathcal{V}_{\mathrm{P}} := \frac{\ker(\mathsf{K}^*|_{\Gamma_{\mathsf{G}}})}{\mathrm{ran}(\mathsf{P}|_{\Gamma_{\mathrm{c}}})} \xrightarrow[\cong]{[\rho_1 \mathsf{G}_1]} \mathcal{V}_\Sigma := \frac{\ker(\mathsf{K}^\dagger_\Sigma|_{\Gamma_{\mathsf{H}}})}{\mathrm{ran}(\mathsf{K}_\Sigma|_{\Gamma_{\mathrm{c}}})} \xrightarrow[\cong]{\mathsf{T}_\Sigma} \mathcal{V}_{\mathsf{R}} := \ker(\mathsf{K}^\dagger_\Sigma|_{\Gamma_{\mathsf{H}}}) \cap \ker(\mathsf{R}^\dagger_\Sigma|_{\Gamma_{\mathsf{H}}})$$

where:

- $\mathsf{R}^\dagger_\Sigma \colon \Gamma(\mathsf{V}_{\rho_1}) \to \Gamma(\mathsf{V}_{\rho_1})$, $\quad \mathsf{R}^\dagger_\Sigma(a_0, \pi_0, a_\Sigma, \pi_\Sigma) := (0, 0, a_\Sigma, \pi_\Sigma)$
- $\Gamma_{\mathsf{G}}(\mathsf{V}_1) := \mathsf{G}_1^{-1}(\Gamma_{\mathsf{R}}(\mathsf{V}_1))$
- $\Gamma_{\mathsf{H}}(\mathsf{V}_{\rho_1}) \subset \Gamma_{\mathrm{c}}(\mathsf{V}_{\rho_1})$ subspace of initial data $(a_0, \pi_0, a_\Sigma, \pi_\Sigma)$ s.t. $a_\Sigma \in \Omega^1_{\mathsf{H}}(\Sigma)$.

Phase Spaces and Complete Gauge Fixing

We call *space of radiation k-forms* $\Omega^k_{\mathsf{R}}(\mathsf{M})$ the subspace of $\Gamma_{\mathrm{sc}}(\mathsf{V}_1) = \Omega^k_{\mathrm{sc}}(\mathsf{M};\mathbb{C})$ defined by

$$\Gamma_{\mathsf{R}}(\mathsf{V}_k) := \begin{cases} \{\omega \in \Omega^k_{\mathrm{sc}}(\mathsf{M};\mathbb{C}) \,|\, \omega_\Sigma|_\Sigma \in \Omega^k_{\mathsf{H}}(\Sigma) \otimes_{\mathbb{R}} \mathbb{C}\} & \text{for } k > 0 \\ C^\infty_{\mathrm{sc}}(\mathsf{M};\mathbb{C}) & \text{for } k = 0 \end{cases},$$

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<u>**Note**</u>: T_Σ represents the complete gauge fixing on the level of initial data.

Construction of Hadamard States I: The Projector T_Σ

By the standard deformation argument, we assume

$$(\mathsf{M}, g) \qquad \text{to be ultrastatic and of bounded geometry}\,.$$

In this case, the phase space of initial data in the gauge (CR) is given by

$$\boxed{\mathcal{V}_\mathsf{R} = \{(a_0, \pi_0, a_\Sigma, \pi_\Sigma) \in \Gamma_\mathsf{H}(\mathsf{V}_{\rho_1}) \mid \delta_\Sigma a_\Sigma = \delta_\Sigma \pi_\Sigma = 0\}\,.}$$

$\Rightarrow$ *How does the projector* $\mathsf{T}_\Sigma := \mathbb{1} - \mathsf{K}_\Sigma(\mathsf{R}_\Sigma \mathsf{K}_\Sigma)^{-1}\mathsf{R}_\Sigma$ *look like in this case?*

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Observation: There is a well-defined projector $\Omega^1_{\mathsf{H}}(\Sigma) = \mathrm{ran}(\mathrm{d}_\Sigma|_{C^\infty_{\mathrm{c}}}) \oplus \ker(\delta_\Sigma) \to \ker(\delta_\Sigma)$:

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Proposition

The projection $\mathsf{T}_\Sigma \colon \ker(\mathsf{K}^\dagger_\Sigma|_{\Gamma_{\mathsf{H}}}) \to \ker(\mathsf{K}^\dagger_\Sigma|_{\Gamma_{\mathsf{H}}})$ is given by

$$\mathsf{T}_\Sigma = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \pi_\delta & 0 \\ 0 & 0 & 0 & \pi_\delta \end{pmatrix}$$

Furthermore, $\ker(\mathsf{T}_\Sigma) = \mathrm{ran}(\mathsf{K}^\dagger_\Sigma|_{\Gamma_{\mathsf{H}}})$ and $\mathrm{ran}(\mathsf{T}_\Sigma) = \mathcal{V}_{\mathsf{R}}$. Hence, $\mathsf{T}_\Sigma \colon \mathcal{V}_\Sigma \to \mathcal{V}_{\mathsf{R}}$ is well-defined.

Note: T_Σ extentible to $\mathsf{T}_\Sigma \colon L^2(\mathsf{V}_{\rho_1}) \to L^2(\mathsf{V}_{\rho_1})$ with $L^2(\mathsf{V}_{\rho_1})$ ... smooth L^2 initial data.

Construction of Hadamard States II: Covariances

⇒ Shubin's ΨDO calculus on manifolds of bounded geometry (Shubin 1992):

↪ Construct invertible square roots ε_i of $\overrightarrow{\Delta}_i$ (Gérard-Murro-Wrochna 2022).

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Note: Since $\mathsf{D}_i = (\partial_t + \epsilon_i)(\partial_t - \epsilon_i)$ modulo smoothing, then

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Theorem

$c^{\pm} := \mathsf{T}_\Sigma \pi^{\pm} \mathsf{T}_\Sigma$ are Cauchy covariances of a Hadamard state on $\mathcal{V}_{\mathrm{P}}$.

Proof *(Sketch).*

- ↪ $\pi^{\pm}$ maps $\Gamma^{\infty}_{\mathsf{H}}(\mathsf{V}_{\rho_1})$ onto $L^2(\mathsf{V}_{\rho_1})$, composition well-defined!
- ↪ $(c^+ + c^-)\mathfrak{f} = T_\Sigma^2 \mathfrak{f} = T_\Sigma \mathfrak{f} = \mathfrak{f}$ mod $\mathrm{ran}(\mathsf{K}_\Sigma|_{\Gamma_\mathsf{H}})$ for $\mathfrak{f} \in \ker(\mathsf{K}_\Sigma^\dagger|_{\Gamma_\mathsf{H}})$
- ↪ Positivity: $\pm\sigma_\Sigma(\mathfrak{f}, c^{\pm} f) = \pm\sigma_\Sigma(\mathfrak{f}, \mathsf{T}_\Sigma\pi^{\pm}\mathsf{T}_\Sigma f) = \pm\sigma_\Sigma(\mathsf{T}_\Sigma\mathfrak{f}, \pi^{\pm}\mathsf{T}_\Sigma f) \geq 0$
- ↪ Hadamard property since T_Σ commutes with $\pi^{\pm}$ up to smoothing. ■

What We Have Seen...

- ↪ Cauchy radiation gauge provides complete gauge fixing and makes fibre metric positive.
- ↪ Complete gauge fixing allows to define positive Hadamard states in the usual way.
- ↪ ΨDO-Projector T_Σ allows to pull back to the space of gauge-invariant observables.

Conclusion and Outlook

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- Apply similar strategy to **Linearized Gravity**:
  - ↪ Possible gauge choices: *de-Donder* or *TT-gauge* with a *Cauchy synchronous gauge*.
  - ↪ Construction of T_Σ more challenging from the technical point of view.
  - ↪ No deformation argument for gravity! Need to construct $c^\pm$ in the general case.

Appendix I: Phase Space of Linear Gauge Theories

Let $\mathsf{P}\colon \Gamma(\mathsf{V}) \to \Gamma(\mathsf{V})$ be a self-adjoint linear differential operator on a Hermitian bundle $(\mathsf{V}, \langle\cdot,\cdot\rangle)$.

- **Linear Observables**: $\Gamma_{\mathrm{c}}(\mathsf{V}) \ni s \mapsto \mathcal{O}_s$ where $\mathcal{O}_s\colon \Gamma(\mathsf{V}) \to \mathbb{C}$ defined by

$$\mathcal{O}_s(\varphi) := \int_{\mathsf{M}} \langle s, \varphi\rangle_{\mathsf{V}} \,\mathrm{vol}_g \,.$$

  $\hookrightarrow$ The assignment $s \mapsto \mathcal{O}_s$ is injective! $\Rightarrow$ Linear observables $\Leftrightarrow$ $\Gamma_{\mathrm{c}}(\mathsf{V})$.

- **Including Dynamics**: $\mathcal{O}_s|_{\ker(\mathsf{P}|_{\Gamma_{\mathrm{sc}}})}$ no longer faithfully labelled by s!

  $\hookrightarrow$ $s, t \in \Gamma_{\mathrm{c}}(\mathsf{V})$ induce same observable on $\ker(\mathsf{P}|_{\Gamma_{\mathrm{sc}}})$ if and only if $s - t \in \mathrm{ran}(\mathsf{P}|_{\Gamma_{\mathrm{c}}})$.

  $\Rightarrow$ Linear on-shell observables $\Leftrightarrow$ $\dfrac{\Gamma_{\mathrm{c}}(\mathsf{V})}{\mathrm{ran}(\mathsf{P}|_{\Gamma_{\mathrm{c}}})}$.

- **Gauge Invariance**: We want those observables for which

$$\mathcal{O}_s(\varphi + \mathsf{K}\omega) = \mathcal{O}_s(\varphi) \qquad \forall\omega$$

  or equivalently $0 = \mathcal{O}_s(\mathsf{K}\omega) = \mathcal{O}_{\mathsf{K}^* s}(\omega)$ for all ω. In other words, $\mathsf{K}^* s \overset{!}{=} 0$.

  $\Rightarrow$ Linear on-shell and gauge-invariant observables $\Leftrightarrow$ $\dfrac{\ker(\mathsf{K}^*|_{\Gamma_{\mathrm{c}}})}{\mathrm{ran}(\mathsf{P}|_{\Gamma_{\mathrm{c}}})}$.

Appendix II: Poisson Equation in the L^2-Setting

Let $\omega \in L^2(\mathsf{T}^*\Sigma) \cap \Omega^1(\Sigma)$. Need to find $f \in C^\infty(\Sigma)$ s.t. $\overrightarrow{\Delta}_0 f = \delta_\Sigma \omega$.

Existence:

Hodge-Kodaira: $L^2(\mathsf{T}^*\mathsf{M}) \cong \overline{\mathrm{d}_\Sigma C_c(\Sigma)} \oplus \overline{\delta_\Sigma \Omega_c^2(\Sigma)} \oplus \ker(\overrightarrow{\Delta}_1|_{L^2})$ $(*)$

$\omega = \alpha + \beta + \gamma$

↪ If $\omega \in \Omega^1(\Sigma)$, then α, β, γ are smooth individually, by elliptic regularity:

$$\begin{aligned} &(i) && (\mathrm{d}_\Sigma + \delta_\Sigma)\alpha = \delta_\Sigma \omega \\ &(ii) && (\mathrm{d}_\Sigma + \delta_\Sigma)\beta = \mathrm{d}_\Sigma \omega \\ &(iii) && \overrightarrow{\Delta}_1 \gamma = 0 \end{aligned}$$

↪ Using *Poincaré duality*, forms in $\Omega^1(\Sigma) \cap \overline{\mathrm{d}_\Sigma C_c(\Sigma)}$ are exact, i.e. $\exists f \in C^\infty(\Sigma)$ s.t. $\alpha = \mathrm{d}_\Sigma f$.

↪ $\omega = \mathrm{d}_\Sigma f + (\beta + \gamma)$ where $\beta + \gamma \in \ker(\delta_\Sigma)$.

Uniqueness:

Let $f \in \mathcal{D}$, i.e. $f \in C^\infty(\Sigma)$ such that $\mathrm{d}_\Sigma f \in \overline{\mathrm{d}_\Sigma C_c(\Sigma)}$.

↪ If $\overrightarrow{\Delta}_0 f = 0$, then $\omega := \mathrm{d}_\Sigma f \in L^2(\mathsf{T}^*\Sigma)$ is closed and co-closed and hence $\omega \in \ker(\overrightarrow{\Delta}_1|_{L^2})$.

↪ Hence, $\omega \in \ker(\overrightarrow{\Delta}_1|_{L^2})$ and $\omega \in \overline{\mathrm{d}_\Sigma C_c(\Sigma)}$ $\Rightarrow$ By $(*)$, $\omega = 0$ and hence $f = \mathrm{const.}$